CHAPTER 3

MATHEMATICAL TOOLS FOR DIGITAL TRANSMISSION ANALYSIS

3.1 INTRODUCTION

The study of digital wireless transmission is in large measure the study of (a) the conversion in a transmitter of a binary digital signal (often referred to as a *baseband* signal) to a modulated RF signal, (b) the transmission of this modulated signal from the transmitter through the atmosphere, (c) the corruption of this signal by noise, unwanted signals, and propagation anomalies, (d) the reception of this corrupted signal by a receiver, and (e) the recovery in the receiver, as best as possible, of the original baseband signal. In order to analyze such transmission, it is necessary to characterize mathematically, in the time, frequency, and probability domains, baseband signals, modulated RF signals, noise, propagation anomalies, and signals corrupted by noise, unwanted signals, and propagation anomalies. The purpose of this chapter is to review briefly the more prominent of those analytical tools used in such characterization—namely, spectral analysis and relevant statistical methods. Spectral analysis permits the characterization of signals in the frequency domain and provides the relationship between frequency domain and time domain characterizations. Noise and propagation anomalies are random processes leading to uncertainty in the integrity of a recovered signal. Thus no definitive determination of the recovered signal can be made. By employing statistical methods, however, computation of the fidelity of the recovered baseband signal is possible in terms of the probability that it's in error. The study of the basic principles of fixed wireless modulation, covered in Chapter 4, will apply several of the tools presented here. Those readers familiar with these tools may want to skip this chapter and proceed to Chapter 4.

3.2 SPECTRAL ANALYSIS OF NONPERIODIC FUNCTIONS

A nonperiodic function of time is a function that is nonrepetitive over time. A stream of binary data as typically transmitted by digital communication systems is a stream of nonperiodic functions, each pulse having equal probability of being one or zero, independent of the value of other pulses in the stream. The analysis of the spectral properties of nonperiodic functions is thus an important component of the study of digital transmission.

3.2.1 The Fourier Transform

A nonperiodic waveform, $v(t)$ say, may be represented in terms of its frequency characteristics by the following relationship:

$$v(t) = \int_{-\infty}^{\infty} V(f) e^{j2\pi ft} \, df \tag{3.1}$$

The factor $V(f)$ is the *amplitude spectral density* or the *Fourier transform*[1] of $v(t)$. It is given by

$$V(f) = \int_{-\infty}^{\infty} v(t) e^{-j2\pi ft} \, dt \tag{3.2}$$

Because $V(f)$ extends from $-\infty$ to $+\infty$ (i.e., it exists on both sides of the zero frequency axis) it is referred to as a *two-sided* spectrum.

An example of the application of the Fourier transform that is useful in the study of digital communications is its use in determining the spectrum of a nonperiodic pulse. Consider a pulse $v(t)$ shown in Fig. 3.1(a), of amplitude V, and that extends from $t = -\tau/2$ to $t = \tau/2$. Its Fourier transform, $V(f)$, is given by

$$\begin{aligned} V(f) &= \int_{-\tau/2}^{\tau/2} V e^{-j2\pi ft} \, dt \\ &= \frac{V}{-j2\pi f}\left[e^{-j2\pi f\tau/2} - e^{j2\pi f\tau/2}\right] \\ &= V\tau \frac{\sin \pi f\tau}{\pi f\tau} \end{aligned} \tag{3.3}$$

The form $(\sin x)/x$ is well known and referred to as the *sampling function, Sa(x)*.[1] The plot of $V(f)$ is shown in Fig. 3.1(b). It will be observed that it is a continuous function. This is a common feature of the spectrum of all nonperiodic waveforms. We note also that it has zero crossings at $\pm 1/\tau$, $\pm 2/\tau$, ….

The Fourier transform $V(f)$ of an impulse of unit strength is also a useful result. By definition an impulse $\delta(t)$ has zero value except at time $t = 0$, and an impulse of unit strength has the property

$$\int_{-\infty}^{\infty} \delta(t) \, dt = 1 \tag{3.4}$$

Thus

$$V(f) = \int_{-\infty}^{\infty} \delta(t) e^{-2\pi jft} \, dt = 1 \tag{3.5}$$

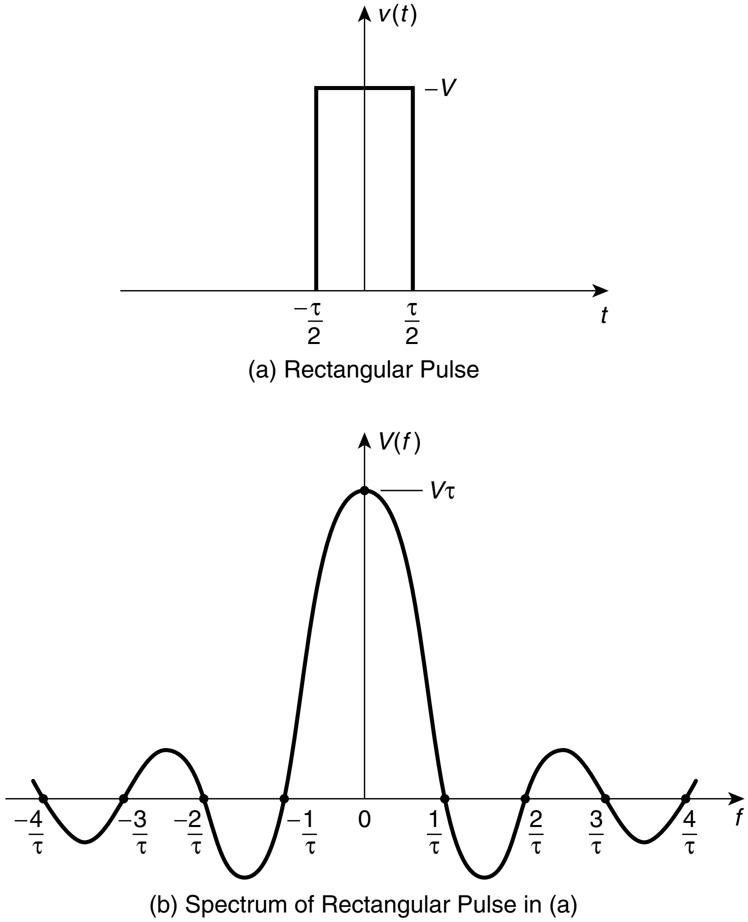


FIGURE 3.1 Rectangular pulse and its spectrum.

Equation (3.5) indicates that the spectrum of an impulse $\delta(t)$ has a constant amplitude and phase and extends from $-\infty$ to $+\infty$.

A final example of the use of the Fourier transform is the analysis of what results in the frequency domain when a signal $m(t)$, with Fourier transform $M(f)$, is multiplied by a sinusoidal signal of frequency f_c. In the time domain the resulting signal is given by

$$\begin{aligned} v(t) &= m(t)\cos 2\pi f_c t \\ &= m(t)\left[\frac{e^{j2\pi f_c t} + e^{-j2\pi f_c t}}{2}\right] \end{aligned} \tag{3.6}$$

and its Fourier transform is thus

$$V(f) = \frac{1}{2}\int_{-\infty}^{\infty} m(t)e^{-j2\pi(f+f_c)t}\,dt + \frac{1}{2}\int m(t)e^{-j2\pi(f-f_c)t}\,dt \tag{3.7}$$

Recognizing that

$$M(f) = \int_{-\infty}^{\infty} m(t) e^{-j2\pi ft}\, dt \tag{3.8}$$

then

$$V(f) = \frac{1}{2} M(f + f_c) + \frac{1}{2} M(f - f_c) \tag{3.9}$$

An amplitude spectrum $|M(f)|$, band limited to the range $-f_m$ to $+f_m$, is shown in Fig. 3.2(a). In Fig.3.2(b), the corresponding amplitude spectrum of $|V(f)|$ is shown.

3.2.2 Linear System Response

A linear system is one in which, in the frequency domain, the output amplitude at a given frequency bears a fixed ratio to the input amplitude at that frequency and the output phase at that frequency bears a fixed difference to the input phase at that frequency, irrespective

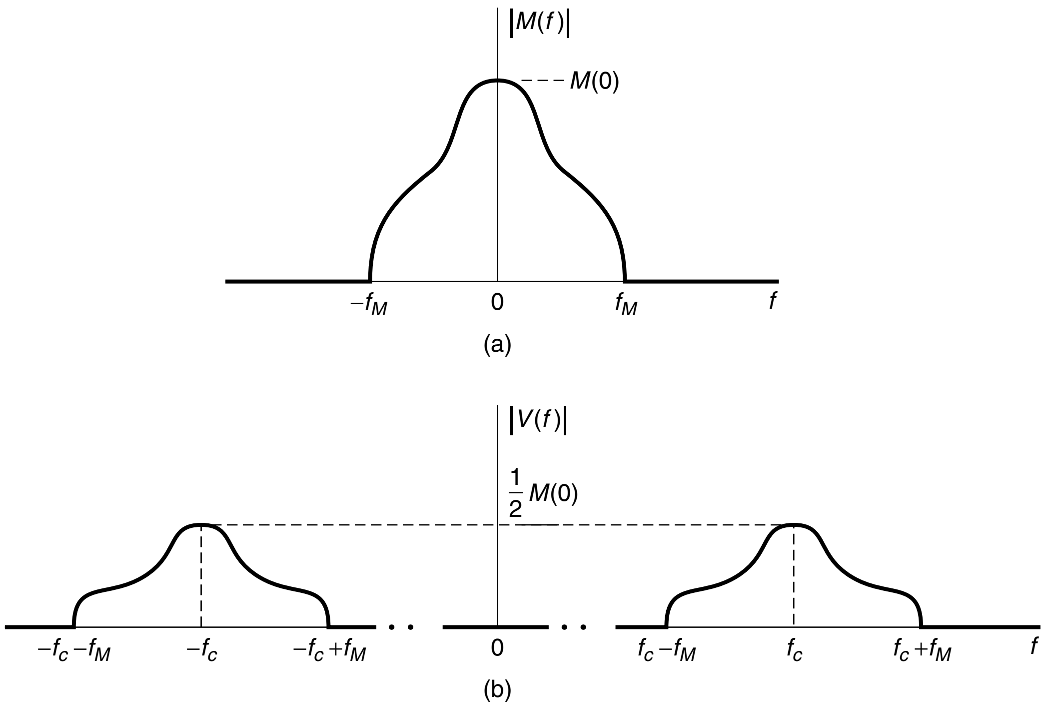


FIGURE 3.2 (a) The amplitude spectrum of a waveform with no special component beyond f_m. (b) The amplitude spectrum of the waveform in (a) multiplied by cos $2\pi f_c t$. (From Taub, H., and Schilling, D., *Principles of Communication Systems,* McGraw-Hill, 1971, and reproduced with the permission of the McGraw-Hill Companies.)

of the absolute value of the input signal. Such a system can be characterized by the complex transfer function, $H(f)$ say, given by

$$H(f) = |H(f)|e^{-j\theta(2\pi f)} \tag{3.10}$$

where $|H(f)|$ represents the absolute amplitude characteristic, and $\theta(2\pi f)$ the phase characteristic of $H(f)$.

Consider a linear system with complex transfer function $H(f)$, as shown in Fig.3.3, with an input signal $v_i(t)$, an output signal $v_o(t)$, and with corresponding spectral amplitude densities of $V_i(f)$, and $V_o(f)$. After transfer through the system, the spectral amplitude density of $V_i(f)$ will be changed to $V_i(f)\ H(f)$. Thus

$$V_o(f) = V_i(f)H(f) \tag{3.11}$$

and

$$v_o(t) = \int_{-\infty}^{\infty} V_i(f)H(f)e^{j2\pi ft}\,df \tag{3.12}$$

An informative situation is the one where the input to a linear system is an impulse function of unit strength. For this case, as per Eq. (3.5), $V_i(f) = 1$, and

$$V_o(f) = H(f) \tag{3.13}$$

Thus, the output response of a linear system to a unit strength impulse function is the transfer function of the system.

3.2.3 Energy and Power Analysis

In considering energy and power in communication systems, it is often convenient to assume that the energy is dissipated in a 1-ohm resistor, as with this assumption one need not keep track of the impact of the true resistance value, R say. When this assumption is made, we refer to the energy as the *normalized energy* and to the power as *normalized power*. It can be shown that the normalized energy E of a nonperiodic waveform $v(t)$, with a Fourier transform $V(f)$, is given by

$$E = \int_{-\infty}^{\infty} [v(t)]^2\,dt = \int_{-\infty}^{\infty} |V(f)|^2\,df \tag{3.14}$$

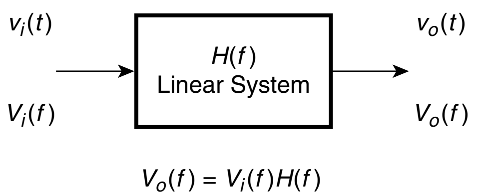


FIGURE 3.3 Signal transfer through a linear system.

The preceding relationship is called *Parseval's theorem.*[1] Should the actual energy be required, then it is simply E [as given in Eq. (3.14)] divided by R.

The *energy density,* $D_e(f)$, of a waveform is the factor $dE(f)/df$. Thus, by differentiating the right-hand side of Eq. (3.14), we have

$$D_e(f) = \frac{dE(f)}{df} = \left|V(f)\right|^2 \tag{3.15}$$

For a nonperiodic function such as a single pulse, normalized energy is finite, but power, which is energy per unit time, approaches zero. Power is thus somewhat meaningless in this context. However, a train of binary nonperiodic adjacent pulses does have meaningful average normalized power. This power, P say, is equal to the normalized energy per pulse E, multiplied by f_s, the number of pulses per second; that is,

$$P = Ef_s \tag{3.16}$$

If the duration of each pulse is τ, then $f_s = 1/\tau$. Substituting this relationship and Eq. (3.14) into Eq. (3.16), we get

$$P = \tfrac{1}{\tau}\int_{-\infty}^{\infty} \left|V(f)\right|^2 df \tag{3.17}$$

The *power spectral density*, $G(f)$, of a waveform is the factor $dP(f)/df$. Thus, by differentiating the right-hand side of Eq. (3.17), we have

$$G(f) = \frac{dP(f)}{df} = \frac{1}{\tau}\left|V(f)\right|^2 \tag{3.18}$$

To determine the effect of a linear transfer function $H(f)$ on normalized power, we substitute Eq. (3.11) into Eq. (3.17). From this substitution we determine that the normalized power, P_o, at the output of a linear network, is given by

$$P_o = \frac{1}{\tau}\int_{-\infty}^{\infty} \left|H(f)\right|^2 \left|V_i(f)\right|^2 df \tag{3.19}$$

Also, from Eq. (3.11), we have

$$\frac{\left|V_o(f)\right|^2}{\tau} = \frac{\left|V_i(f)\right|^2}{\tau}\left|H(f)\right|^2 \tag{3.20}$$

Substituting Eq. (3.18) into Eq. (3.20), we determine that the power spectral density $G_o(f)$ at the output of a linear network is related to the power spectral density $G_i(f)$ at the input by the relationship

$$G_o(f) = G_i(f)\left|H(f)\right|^2 \tag{3.21}$$

3.3 STATISTICAL METHODS

We now turn our attention away from the time and frequency domain and toward the probability domain where statistical methods of analysis are employed. As indicated in Section 3.1, such methods are required because of the uncertainty resulting from the introduction of noise and other factors during transmission.

3.3.1 The Cumulative Distribution Function and the Probability Density Function

A *random variable* X[1,2] is a function that associates a unique numerical value $X(\lambda_i)$ with every outcome λ_i of an event that produces random results. The value of a random variable will vary from event to event, and depending on the nature of the event will be either *discrete* or *continuous*. An example of a discrete random variable X_d is the number of heads that occur when a coin is tossed four times. As X_d can only have the values 0, 1, 2, 3, and 4, it is discrete. An example of a continuous random variable X_c is the distance of a shooter's bullet hole from the bull's eye. As this distance can take any value, X_c is continuous.

Two important functions of a random variable are the *cumulative distribution function* (*CDF*) and the *probability density function* (*PDF*).

The cumulative distribution function, $F(x)$, of a random variable X is given by

$$F(x) = P\left[X(\lambda) \le x\right] \tag{3.22}$$

where $P[X(\lambda) \le x]$ is the probability that the value $X(\lambda)$ taken by the random variable X is less than or equal to the quantity x.

The cumulative distribution function $F(x)$ has the following properties:

1. $0 \le F(x) \le 1$
2. $F(x_1) \le F(x_2)$ if $x_1 \le x_2$
3. $F(-\infty) = 0$
4. $F(+\infty) = 1$

The *probability density function* $f(x)$ of a random variable X is the derivative of $F(x)$ and thus is given by

$$f(x) = \frac{dF(x)}{dx} \tag{3.23}$$

The probability density function $f(x)$ has the following properties:

1. $f(x) \ge 0$ for all values of x
2. $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Further, from Eqs. (3.22) and (3.23), we have

$$F(x) = \int_{-\infty}^{x} f(z)\, dz \tag{3.24}$$

The function within the integral is not shown as a function of x because, as per Eq. (3.22), x is defined here as a fixed quantity. It has been arbitrarily shown as a function of z, where z has the same dimension as x, $f(z)$ being the same PDF as $f(x)$. Some texts, however, show it equivalently as a function of x, with the understanding that x is used in the generalized sense.

The following example will help in clarifying the concepts behind the PDF, $f(x)$, and the CDF, $F(x)$. In Fig. 3.4(a) a four-level pulse amplitude modulated signal is shown. The amplitude of each pulse is random and equally likely to occupy any of the four levels. Thus, if a random variable X is defined as the signal level v, and $P(v = x)$ is the probability that $v = x$, then

$$P(v = -3) = P(v = -1) = P(v = +1) = P(v = +3) = 0.25 \tag{3.25}$$

With this probability information we can determine the associated CDF, $F_{4L}(v)$. For example, for $v = -1$

$$F_{4L}(-1) = P(v \leq -1) = P(v = -3) + P(v = -1) = 0.5 \tag{3.26}$$

In a similar fashion, $F_{4L}(v)$ for other values of v may be determined. A plot of $F_{4L}(v)$ versus v is shown in Fig. 3.4(b).

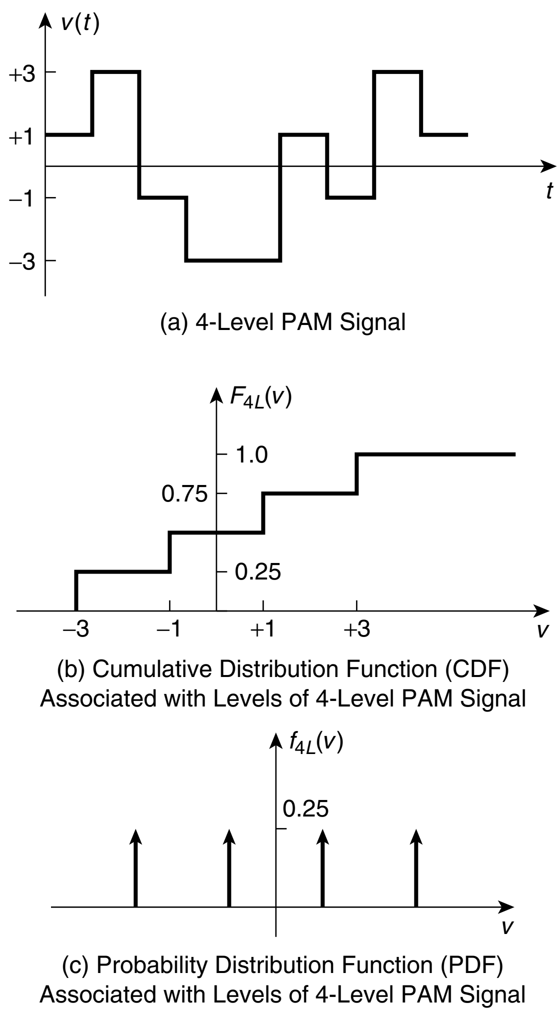


FIGURE 3.4 A four-level PAM signal and its associated CDF and PDF.

The PDF $f_{4L}(v)$ corresponding to $F_{4L}(v)$ can be found by differentiating $F_{4L}(v)$ with respect to v. The derivative of a step of amplitude V is a pulse of value V. Thus, since the steps of $F_{4L}(v)$ are of value 0.25,

$$f_{4L}(-3) = f_{4L}(-1) = f_{4L}(+1) = f_{4L}(+3) = 0.25 \tag{3.27}$$

A plot of $f_{4L}(v)$ versus v is shown in Fig. 3.4(c).

3.3.2 The Average Value, the Mean Squared Value, and the Variance of a Random Variable

The *average value* or *mean, m,* of a random variable X, also called the *expectation* of X, is also denoted either by $\overline{X}$ or $E(x)$. For a discrete random variable, X_d, where n is the total number of possible outcomes of values $x_1, x_2, \ldots, x_n$, and where the probabilities of the outcomes are $P(x_1), P(x_2), \ldots, P(x_n)$ it can be shown that

$$m = \overline{X_d} = E(X_d) = \sum_{i=1}^{n} x_i P(x_i) \tag{3.28}$$

For a continuous random variable X_c, with PDF $f_c(x)$, it can be shown that

$$m = \overline{X_c} = E(X_c) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx \tag{3.29}$$

and that the *mean square value,* $\overline{X_c^2}$ or $E(X_c^2)$ is given by

$$\overline{X_c^2} = E(X_c^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx \tag{3.30}$$

Figure 3.5 shows an arbitrary PDF of a continuous random variable. A useful number to help in evaluating a continuous random variable is one that gives a measure of how widely spread its values are around its mean m. Such a number is the root mean square (rms) value of $(X - m)$ and is called the *standard deviation* σ of X.

The square of the standard deviation, σ^2, is called the variance of X and is given by

$$\sigma^2 = E\left[(X-m)^2\right] = \int_{-\infty}^{\infty} (x-m)^2 f(x)dx \tag{3.31}$$

The relationship between the variance σ^2 and the mean square value $E(X^2)$ is given by

$$\begin{aligned} \sigma^2 &= E\left[(X-m)^2\right] \\ &= E\left[X^2 - 2mX + m^2\right] \\ &= E\left(X^2\right) - 2mE(X) + m^2 \\ &= E\left(X^2\right) - m^2 \end{aligned} \tag{3.32}$$

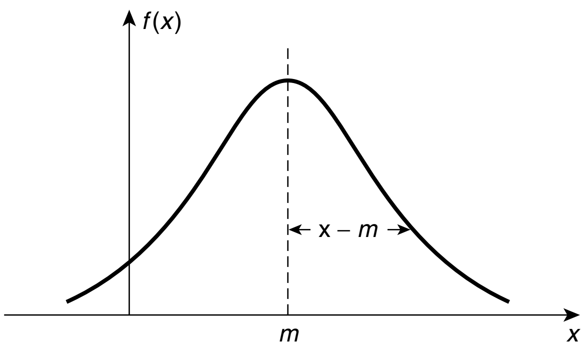


FIGURE 3.5 A Probability Distribution Function (PDF) of a continuous random variable.

We note that for the average value $m = 0$, the variance $\sigma^2 = E(X^2)$.

3.3.3 The Gaussian Probability Density Function

The *Gaussian* or, as it's sometimes called, the *normal* PDF[1,2] is very important to the study of wireless transmission and is the function most often used to describe *thermal noise*. Thermal noise is the result of thermal motions of electrons in the atmosphere, resistors, transistors, and so on and is thus unavoidable in communication systems. The Gaussian probability density function, $f(x)$, is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2} \tag{3.33}$$

where m is as defined in Eq. (3.28) and σ as defined in Eq. (3.31). When $m = 0$ and $\sigma = 1$ the *normalized Gaussian probability density function* is obtained. A graph of the Gaussian PDF is shown in Fig. 3.6(a).

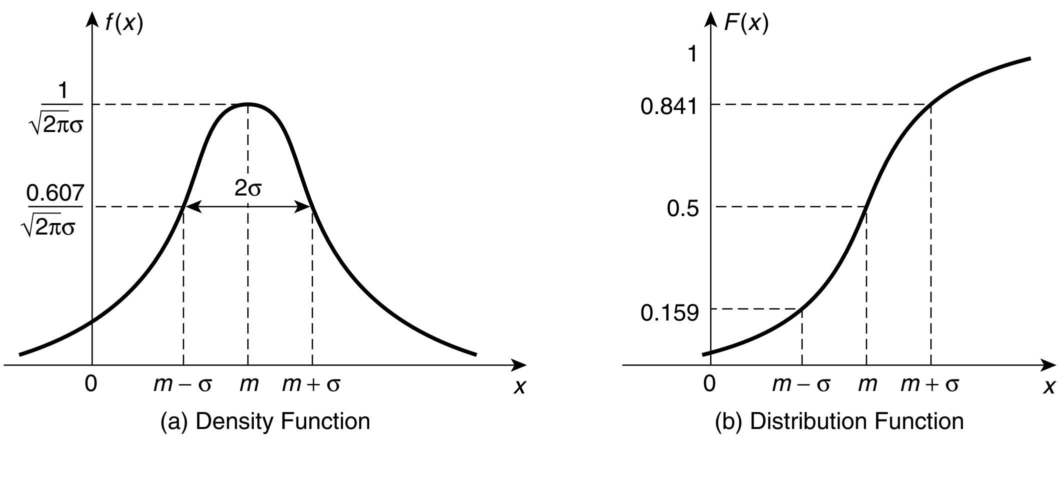


FIGURE 3.6 The Gaussian random variable.

The CDF corresponding to the Gaussian PDF is given by

$$F(x) = P\left[X(\lambda) \le x\right] = \int_{-\infty}^{x} \frac{e^{-(z-m)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} dz \tag{3.34}$$

When $m = 0$, the *normalized Gaussian cumulative distribution function* is obtained and is given by

$$F(x) = \int_{-\infty}^{x} \frac{e^{-z^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} dz \tag{3.35}$$

A graph of the Gaussian cumulative distribution function is shown in Fig. 3.6(b). In practice, since the integral in Eq. (3.35) is not easily determined, it is normally evaluated by relating it to the well-known and numerically computed function, the *error function.* The error function of v is defined by

$$erf(v) = \frac{2}{\sqrt{\pi}} \int_{0}^{v} e^{-u^2} du \tag{3.36}$$

and it can be shown that $erf(0) = 0$ and $erf(\infty) = 1$.

The function $[1 - erf(v)]$ is referred to as the *complementary error function*, $erfc(v)$. Noting that $\int_0^v = \int_0^\infty - \int_v^\infty$, we have

$$\begin{aligned} \text{erfc}(v) &= 1 - \text{erf}(v) \\ &= 1 - \left[\frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} du - \frac{2}{\sqrt{\pi}} \int_v^\infty e^{-u^2} du\right] \\ &= 1 - \left[erf(\infty) - \frac{2}{\sqrt{\pi}} \int_v^\infty e^{-u^2} du\right] \\ &= \frac{2}{\sqrt{\pi}} \int_v^\infty e^{-u^2} du \end{aligned} \tag{3.37}$$

Tabulated values of $erfc(v)$ are only available for positive values of v.

Using the substitution $u \equiv x/\sqrt{2}\sigma$, it can be shown[1] that the Gaussian CDF $F(x)$ of Eq. (3.35) may be expressed in terms of the complementary error function of Eq. (3.37) as follows:

$$F(x) = 1 - \frac{1}{2} erfc\left(\frac{x}{\sqrt{2}\sigma}\right) \qquad \text{for x} \ge 0 \tag{3.38a}$$

$$= \frac{1}{2} erfc\left(\frac{|x|}{\sqrt{2}\sigma}\right) \qquad \text{for x} \ge 0 \tag{3.38b}$$

3.3.4 The Rayleigh Probability Density Function

The propagation of wireless signals through the atmosphere is often subject to multipath fading. Such fading will be described in detail in Chapter 5. Multipath fading is best characterized by the *Rayleigh* PDF.[1] Other phenomena in wireless transmission are also characterized by the Rayleigh PDF, making it an important tool in wireless analysis. The Rayleigh probability density function $f(r)$ is defined by

$$f(r) = \frac{r}{\alpha^2} e^{-r^2/2\alpha^2}, \qquad 0 \le r \le \infty \tag{3.39a}$$

$$= 0, \qquad r < 0 \tag{3.39b}$$

and hence the corresponding CDF is given by

$$F(r) = P[R(\lambda) \le r] = 1 - e^{-r^2/2\alpha^2}, \qquad 0 \le r \le \infty \tag{3.40a}$$

$$= 0, \qquad r < 0 \tag{3.40b}$$

A graph of $f(r)$ as a function of r is shown in Fig. 3.7. It has a maximum value of $1/(\alpha\sqrt{e})$, which occurs at $r = \alpha$. It has a mean value $\overline{R} = \sqrt{\pi/2} \cdot \alpha$, a mean-square value $\overline{R^2} = 2\alpha^2$, and hence, by Eq. (3.32), a variance σ^2 given by

$$\sigma^2 = \left(2 - \frac{\pi}{2}\right)\alpha^2 \tag{3.41}$$

A graph of $F(r)$ versus $10 \log_{10} (r^2 / 2\alpha^2)$, which is from Feher,[3] is shown in Fig. 3.8. If the amplitude envelope variation of a radio signal is represented by the Rayleigh random variable R, then the envelope has a mean-square value of $\overline{R^2} = 2\alpha^2$, and hence the signal has an average power of $\overline{R^2} / 2 = \alpha^2$. Thus, $10 \log_{10} (r^2 / 2\alpha^2)$, which equals $10 \log_{10} (r^2 / 2) - 10 \log_{10} (\alpha^2)$, represents the decibel difference between the signal power level when its amplitude is r and its average power. From Fig. 3.8 it will be noted that for signal power less than the average power by 10 dB or more, the distribution function $F(r)$ decreases by a factor of 10 for every 10-dB decrease in signal power. As a result, when fading radio signals exhibit this behavior, the fading is described as Rayleigh fading.

3.3.5 Thermal Noise

White noise[1] is defined as a random signal whose power spectral density is constant (i.e., independent of frequency). True white noise is not physically realizable since constant power spectral density over an infinite frequency range implies infinite power. However, thermal noise, which as indicated earlier has a Gaussian PDF, has a power spectral density that is relatively uniform up to frequencies of about 1000 GHz at room temperature (290K), and up to about 100 GHz at 29K.[4] Thus, for the purpose of practical communications

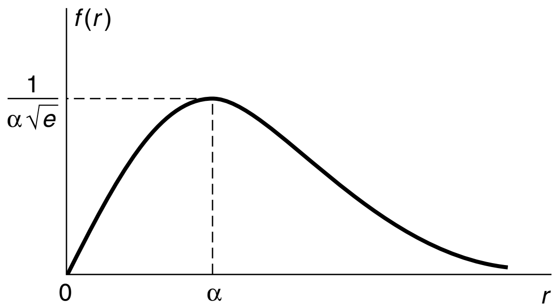


FIGURE 3.7 The Rayleigh probability density function. (From Taub, H., and Schilling, D., *Principles of Communication Systems,* McGraw-Hill, 1971, and reproduced with the permission of the McGraw-Hill Companies.)

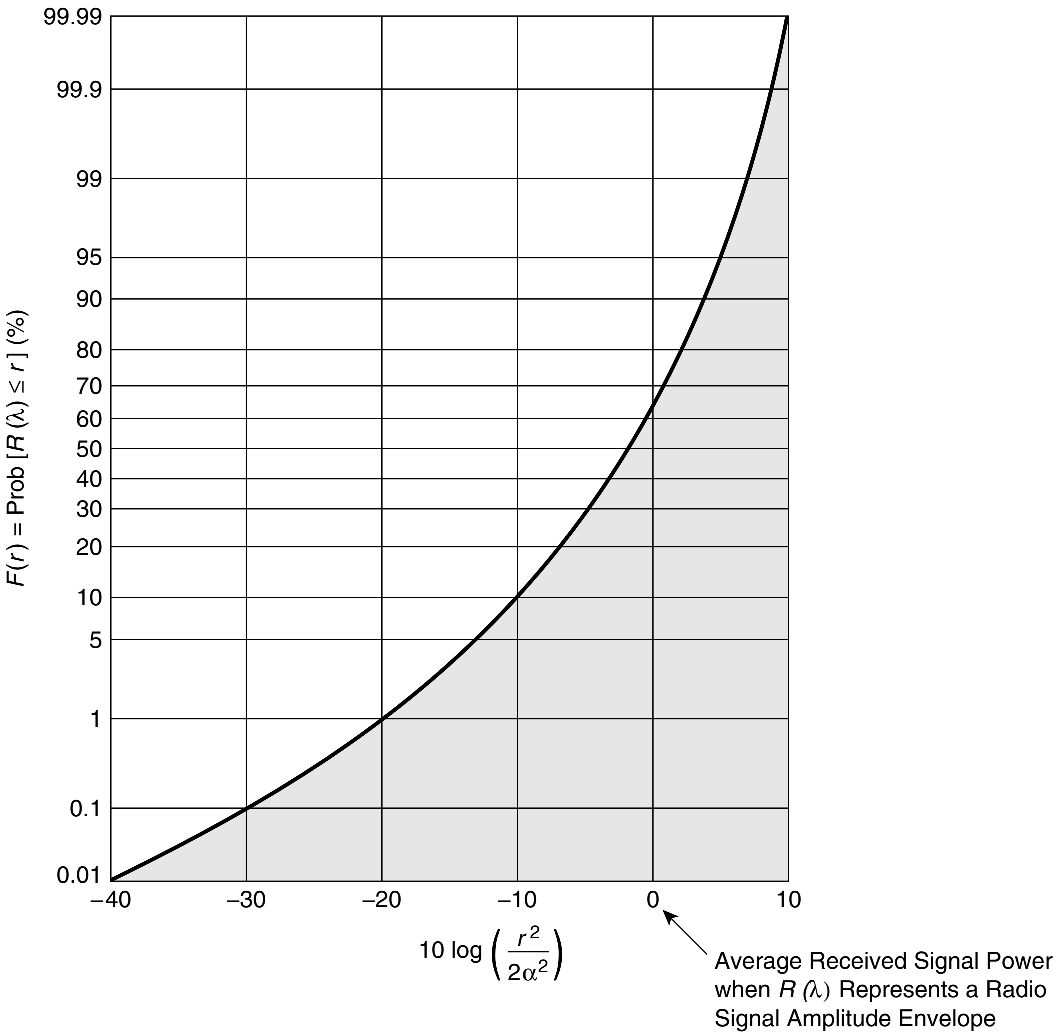


FIGURE 3.8 The Rayleigh cumulative distribution function. (By permission from Ref. 3.)

analysis, it is regarded as white. A simple model for thermal noise is one where the two-sided power spectral density $G_n(f)$ is given by

$$G_n(f) = \frac{N_0}{2} \text{ Watts / hertz} \tag{3.42}$$

where N_0 is a constant.

In a typical wireless communications receiver, the incoming signal and accompanying thermal noise is normally passed through a symmetrical bandpass filter centered at the carrier frequency f_c to minimize interference and noise. The width of the bandpass filter, W, is normally small compared to the carrier frequency. When this is the case the filtered noise can be characterized via its so-called *narrowband representation*.[1] In this representation, the filtered noise voltage, $n_{nb}(t)$, is given by

$$n_{nb}(t) = n_c(t)\cos 2\pi f_c t - n_s(t)\sin 2\pi f_c t \tag{3.43}$$

where $n_c(t)$ *and* $n_s(t)$ are Gaussian random processes of zero mean value, of equal variance and, further, independent of each other. Their power spectral densities, $G_{n_c}(f)$ and $G_{n_s}(f)$, extend only over the range $-W/2$ to $W/2$ and are related to $G_n(f)$ as follows:

$$G_{n_c}(f) = G_{n_s}(f) = 2G_n(f_c + f) \tag{3.44}$$

The relationship between these power spectral densities is shown in Fig. 3.9. This narrowband noise representation will be found to be very useful when we study carrier modulation methods.

3.3.6 Noise Filtering and Noise Bandwidth

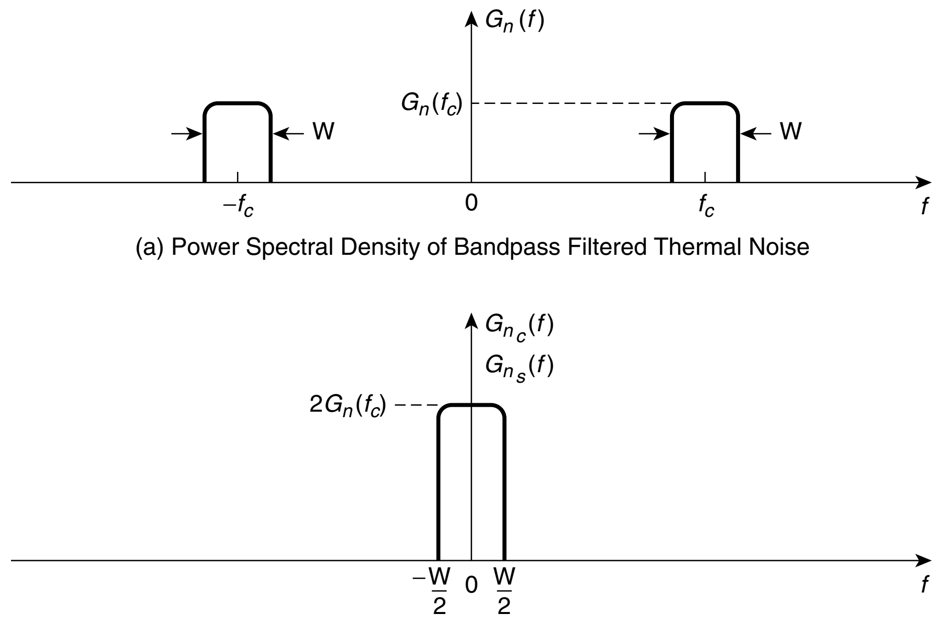
In a receiver, a received signal contaminated with thermal noise is normally filtered to minimize the noise power relative to the signal power prior to demodulation. If, as shown in Fig. 3.10, the input two-sided noise spectral density is $N_0 / 2$, the transfer function of the real filter is $H_r(f)$, and the output noise spectral density is $G_{no}(f)$, then, by Eq. (3.21), we have

$$G_{no}(f) = \frac{N_0}{2}\left|H_r(f)\right|^2 \tag{3.45}$$

and thus the normalized noise power at the filter output, P_o, is given by

$$\begin{aligned} P_o &= \int_{-\infty}^{\infty} G_{no}(f)\, df \\ &= \frac{N_0}{2}\int_{-\infty}^{\infty} \left|H_r(f)\right|^2 df \end{aligned} \tag{3.46}$$

A useful quantity to compare the amount of noise passed by one receiver filter versus another is the filter *noise bandwidth*.[1] The noise bandwidth of a filter is defined as the width of an ideal brick-wall (rectangular) filter that passes the same average power from a white noise source as does the real filter. In the case of a real low pass filter, it is assumed that



(a) Power Spectral Density of Bandpass Filtered Thermal Noise

(b) Power Spectral Density of Narrowband Noise Representation Components n_c and n_s

FIGURE 3.9 Spectral density relationships associated with narrowband representation of noise.

the absolute values of the transfer functions of both the real and brick-wall filters are normalized to one at zero frequency. In the case of a real bandpass filter, it is assumed that the brick-wall filter has the same center frequency as the real filter, f_c say, and that the absolute values of the transfer functions of both the real and brick-wall fitlers are normalized to one at f_c.

For an ideal brick-wall low pass filter of two-sided bandwidth B_n and $|H_{bw}(f)| = 1$ from $-B_n/2$ to $+B_n/2$

$$P_o = \frac{N_0}{2} B_n \tag{3.47}$$

Thus, from Eqs. (3.46) and (3.47) we determine that

$$B_n = \int_{-\infty}^{\infty} |H_r(f)|^2 \, df \tag{3.48}$$

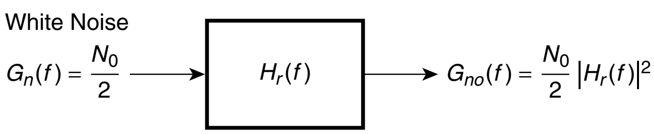


FIGURE 3.10 Filtering of white noise.

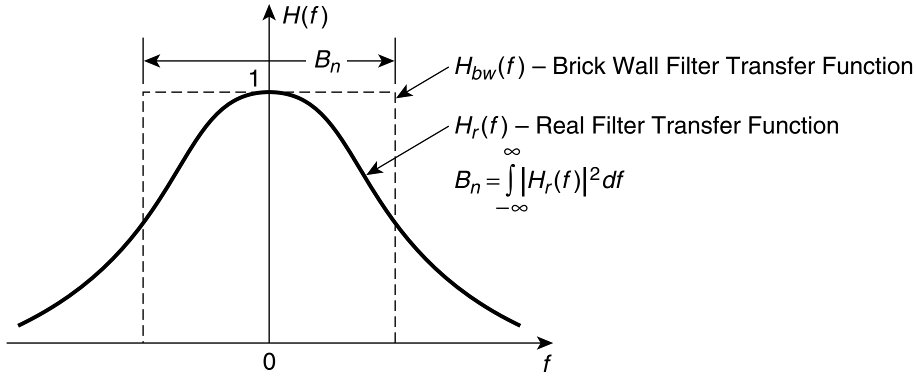


FIGURE 3.11 Low pass filter two-sided noise bandwidth, B_n.

Figure 3.11 shows the transfer function $H_{bw}(f)$ of a low pass brick-wall filter of two-sided noise bandwidth B_n superimposed on the two-sided transfer function $H_r(f)$ of a real filter.

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