Technological Constraints and Economic Growth

Reiner Kümmel,[a] *Robert U. Ayres,*[b] *and Dietmar Lindenberger* [c] *

[a] Institute for Theoretical Physics and Astrophysics, University of Würzburg, D-97074 Würzburg, Germany,

[b] European School of Business Administration INSEAD, Fontainebleau 77305, France, and IIASA, A-2361 Laxenburg, Austria

[c] Institute of Energy Economics, University of Cologne, D-50923 Cologne, Germany.

*Corresponding author; email: dietmar.lindenberger@uni-koeln.de

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Abstract

Optimization of profit or time-integrated utility subject to technological constraints on factor combinations yields new conditions for economic equilibrium: Output elasticities are no longer equal to cost shares. The relationship is more complex, depending on factor prices plus shadow prices. The latter are due to the constraints. We calculate output elasticities and shadow prices in a three-factor model with capital, labor, and energy or useful work. Limits to capacity utilization and automation constrain the inputs. The implications of this new framework for growth theory are discussed.

JEL classifications: A12, C02, E13, Q43

Keywords: Optimization, technological constraints, shadow prices, output elasticities, energy.

I. INTRODUCTION

Systems attain equilibrium when their variables adjust within given constraints so that a system-specific objective becomes an extremum. Correspondingly, economic models involve agents optimizing subject to all relevant constraints. Profit and utility maximization subject to technological constraints on the combinations of capital, labor, and energy, and the consequences for macroeconomic production functions, are the subject of this paper.

The development of economic growth theory since Robert Solow's original work in the 1950s (Solow 1956, 1957), with few exceptions, has assumed a production function with two factors of growth, capital K and labor L. Then the oil price shocks 1973-1975 and 1979-1981 and the accompanying recessions known as the first and the second energy crisis prompted investigations that consider energy E, sometimes in combination with materials M, as an additional factor of production. Early examples are the studies of Hudson and Jorgenson (1974), Griffin and Gregory (1976), Berndt and Jorgenson (1978), Berndt and Wood (1979), Allen (1979), and Jorgenson (1978, 1984).

In most KLE (or $KLEM$) models output elasticities equal factor cost shares, as unconstrained equilibrium requires. Average OECD factor cost shares have been roughly 25% for capital, 70% for labor, and only 5% for energy before the oil price fluctuations that started in 2004. Therefore, the economic role of energy has been considered as marginal or negligible. Denison (1979), for instance, argues: "Energy gets about 5 percent of the total input weight in the business sector . . . If . . . the weight of energy is 5 percent, a 1-percent reduction in energy consumption with no change in labor and capital would reduce output by 0.05 percent." According to this argument, the decrease of energy input in the US sector "Industries" by 7.3 percent between 1973 and 1975 could not possibly have caused the observed reduction of output by 5.3 percent. On the other hand, Jorgenson (1988) concluded "that there was a dramatic impact of energy prices on economic growth during the energy crisis."

Since the cost shares of capital and labor have been very nearly constant for many decades and the cost share of aggregate energy varied by not more than three percent during the first and second energy crises, the Cobb-Douglas production function with constant output elasticities that are assumed to equal factor cost shares, has been used as a convenient tool for analyses of production and growth.[2] This assumption also shows in contributions to endogeneous growth theory. For instance, Barbier (1999) combines Stiglitz' (1974) neoclassical production function, which includes a natural resource, with Romer's (1990) model of endogeneous technological change. He investigates paths of optimal growth, using a production function of the Cobb-Douglas type with constant returns to scale, multiplied by the stock of knowledge. Welsch and Eisenack (2002) start from Romer's model, too, and extend it to examine the impact of secular changes in energy cost on technological progress and long run growth. Their production function is also Cobb-Douglas with constant returns to scale and output elasticities of capital, labor, and energy that result from the equilibrium equations between factor prices and marginal products. There are also models with only two factors like that of Bertola (1993), who studies distributional implications of growth-oriented policies, describing aggregate output by the product of a term representing disembodied productivity with a constant-returns-to-scale Cobb-Douglas production function of aggregate capital K and a factor L that "might refer to land or (uneducated) labor"; the output elasticities are given by the shares of K and L in aggregate income.

Factor-cost-share weighting invariably requires an additional time-dependent multiplier, representing technical progress, in order to account for the Solow residual. Within this framework growth theorists have largely considered energy as a marginal factor, or neglected it altogether, despite econometric evidence of its importance, presented, e.g., by Hamilton (1983, 2003).

"Weighting the Options of Global Warming Policies" by means of the DICE model, Nordhaus (2008) assumes that "Output is produced by a Cobb-Douglas production function in capital, labor, and energy". However, profit maximization subject to technological constraints on factor combinations modifies cost-share weighting of production factors by introducing shadow prices. This is shown in Section II. In Section III we note the technological constraints on the combinations of capital, labor, and energy. There is complementarity between capital vis a vis labor and energy, and capacity utilization cannot exceed 100 percent. Moreover, the possibility of substituting capital and energy

[2]The Cobb-Douglas function can be considered as the zero-order approximation to more general production functions, like the translog function.

for labor is constrained by the technologically feasible degree of automation. We use the term automation hereafter to include both mechanization and computerization. Section IV introduces the Linex production function. This function and its output elasticities are calculated without using equilibrium conditions. We show how the shadow prices can be actually computed with its help. Section V summarizes the results and indicates climate change and resource scenarios the analyses of which may benefit from the modified framework of growth theory. Mathematical details are given in Appendices A, B, and C.

II. SHADOWED COST SHARES

Output elasticities equal factor cost shares in equilibrium conditions that result from profit maximization subject to no constraint except fixed cost[3] under constant returns to scale. However, as noted above, there are also technological constraints. These are limitations on the possible combinations of production factors that are due to the physical properties of the capital stock. For example, there is a limit to the substitution of capital (machines) and energy for labor in the ongoing processes of automation. The feasible degree of automation depends critically on the energy requirements of engines and information processors, and on the volume of the latter as well.

How will technological constraints change the equilibrium conditions and output elasticities? The general answer is: If output Y is produced by N factors of production $X_1 \dots X_i \dots X_N$, whose combinations are subject to a number of technological constraints, labeled by the index a and expressed by the equations $f_a(X_1 \dots X_i \dots X_N, t) = 0$ with the help of slack variables, profit maximization under constant returns to scale results in N equilibrium conditions

$$\epsilon_i \equiv \frac{X_i}{Y}\frac{\partial Y}{\partial X_i} = \frac{X_i\,[p_i + s_i]}{\sum_{i=1}^{N} X_i\,[p_i + s_i]}, \quad i = 1 \dots N \quad . \tag{1}$$

These conditions relate the output elasticities ϵ_i of factors X_i to market prices p_i per factor unit and the factor shadow prices

$$s_i \equiv -\sum_a \frac{\mu_a}{\mu}\frac{\partial f_a}{\partial X_i} \quad , \tag{2}$$

where μ_a and μ are the Lagrange multipliers associated with the technological and fixed-cost constraints, respectively.

Thus, the output elasticities in eq. (1) are equal to what one may call “shadowed” cost shares. Intertemporal utility optimization yields the same,[4] as shown in Appendix B.

[3]Cost are fixed, if output is maximized. Alternatively, output is fixed, if cost are minimized.

[4]For decreasing marginal utility of consumption, $dU/dC < 0$, the shadow price of capital contains an additional term proportional to the time derivative of dU/dC. This term vanishes in linear approximation.

If there were no technological constraints, all Lagrange multipliers μ_a would be zero, the shadow prices s_i would vanish, and one would have the usual factor cost shares on the r.h.s of eq. (1). In the presence of technological constraints and non-zero shadow prices, on the other hand, the ratios μ_a/μ of Lagrange multipliers are finite and functions of the output elasticities ϵ_i. This is explicitly shown in Appendix A for a three factor model, $N = 3$, with two technological constraint equations $f_A(X_1(t), X_2(t), X_3(t), t) = 0$ and $f_B(X_1(t), X_2(t), X_3(t), t) = 0$, by resolving the two independent ratios of output elasticities, ϵ_1/ϵ_2 and ϵ_1/ϵ_3, with respect to μ_A/μ and μ_B/μ. This requires some algebraic manipulations but is otherwise straightforward. The actual calculations of the $\partial f_A/\partial X_i$ and $\partial f_B/\partial X_i$ in eq. (2) from the constraints on capital, labor, and energy are performed in Appendix C.

The constraint of conserving a suitably defined capital stock when maximizing utility is treated by Hellwig et al. (2000) in the context of intertemporal decision problems. Ruttan (2002, 2004) notes constraints in agricultural production, such as diminishing yield increases from incremental fertilizer application and declining reductions in labor input from the use of larger and more powerful mechanical equipment. Furthermore, peak oil (e.g. Strahan, 2007) and climate change (e.g. Stern, 2007, 2008) exemplify resource and environmental constraints on fossil-fuel-based technologies. To keep things simple we leave these constraints out of the present discussion and concentrate on the constraints that result from the basic engineering principles that determine the possible combinations of capital, labor, and energy.

The important point of this section is that output elasticities have to be computed independently from the equilibrium conditions because of the elasticity-dependent shadow prices.

III. TECHNOLOGICAL CONSTRAINTS ON CAPITAL, LABOR, AND ENERGY

Technological constraints are part of production models. In a world where the capital stock consists of simple tools (and sheds to house them) output results from the combination of human and animal muscles[5] with these tools. In such a world the disregard of technological constraints is not unreasonable.

On the other hand, and in contrast to the pre-industrial situation, the production system of an industrialized country *is* subject to binding technological constraints. In order to illustrate this we consider a model where the capital stock $K(t)$ at time t consists of all energy-converting and information-processing machines together with all buildings and installations necessary for their protection and operation. Output $Y(t)$ results from work performance and information processing by the combination of such capital with

[5] In ancient Rome the Latin word for money, "pecunia", originally meant cattle. In much of Africa today wealth is measured in livestock units.

(routine) labor $L(t)$ and energy $E(t)$.[6]

Capital in the absence of energy (and supervision by humans) is functionally inert. Nothing happens. To be productive the machines of the capital stock must be activated by useful energy (or more precisely *exergy* – spelled with an x), for which we use the symbol E. In the engineering sciences exergy is the name for the maximum amount of physical work that can be obtained in principle from a given quantity of energy. Primary energy in the form of coal, oil, gas, nuclear fuels or solar radiation can be converted to nearly 100 percent into physical work under appropriate conditions. Thus, it is practically all exergy. It is an input ultimately supplied by nature.

To be economically productive capital must also be allocated, organized and supervised by (human) labor. Economic activities of humans can be subdivided into two components: 1) routine labor, which (by definition) can be substituted by some combination of capital and energy and 2) a residual that cannot be replaced at any particular moment in time. The latter component could be given various names, but the one we prefer is *creativity*. Creativity, in this sense, is the specific human contribution to production and growth that cannot be provided by any machine, even a sophisticated computer capable of learning from experience. It includes ideas, inventions, valuations, and (especially) interactive decisions depending on human reactions and characteristics. It is important to recognize that the non-routine component of human labor may decline over time, but it is never zero.

The ultimate lower limit of routine labor inputs is probably un-knowable, because it depends to some degree on the limits of artificial intelligence. But we need not concern ourselves with the ultimate limit. At any given time, with a given technology and state of automation, there is a limit to the extent that routine labor can increase output. In other words, the model postulates the possibility of a combination of capital and exergy such that adding one more unskilled worker adds nothing to gross economic output. (In some manufacturing sectors of industrialized countries this point does not seem to be far away.)

There is another fairly obvious technological constraint on the combinations of factors. In brief, machines are designed and built for specific exergy inputs.[7] In some cases

[6]Hudson and Jorgenson (1974) and Berndt and Wood (1975) take materials into account as a fourth factor of production. Since materials are passive partners of the production process, which do not contribute actively to output – their atoms and molecules are merely arranged in orderly patterns by capital, labor and energy when value added is created – we do not include them in the model. This keeps the exercise simple. Like other models the present model also disregards land as a production factor. Land area matters mainly as site for production facilities and for photosynthetic conversion of solar energy into the chemical energy of glucose in agriculture. It does not contribute actively to work performance and information processing.

[7]If the exergy available is less than the design requirement, production will be less than optimal. As an example, the efficiency of electricity generating plants (electricity output divided by fuel input) is reduced when operating in part load.

(e.g. for some electric motors) there is a modest overload capability. Buildings can be over-heated or over-cooled, to be sure, but this does not contribute to productivity. But on average the maximum exergy input is fixed by design. Both energy convertors and energy users have built-in limits. In other words, the ratio of exergy to capital must not exceed a definite upper limit.

The bottom line of the above considerations is that the use of capital, labor and energy in industrial systems is subject to technological constraints that are the consequence of limits to capacity utilization and to the substitution of capital and energy for labor. We use "degree of automation" as a shorthand for this substitution.

For the derivation of the constraint equations output and inputs must be specified by measurement prescriptions. Output Y and capital stock K are measured in deflated monetary units, as reported by the national accounts. Routine labor L is measured in man hours worked per year, as given by the national labor statistics, and energy E is measured in petajoules (or tons of oil equivalents, or quads) per year, as shown by the national energy balances.

Of course, the theory must be independent from the choice of units. Therefore, in our three-factor model, with $X_1 \equiv K, X_2 \equiv L$, and $X_3 \equiv E$, we work with output and inputs normalized to the output Y_0 and the inputs K_0, L_0, and E_0 in a base year t_0. *The new, dimensionless variables are written with lower case letters.* The transformation to normalized capital k, labor l, and energy e at time t is given by

$$k \equiv K/K_0, \quad l \equiv L/L_0, \quad e \equiv E/E_0 \quad , \tag{3}$$

and the normalized production function is

$$y(k, l, e) \equiv Y(kK_0, lL_0, eE_0)/Y_0 \quad . \tag{4}$$

From here on we work in the "space" of dimensionless, normalized inputs and outputs, defined by eqs. (3) and (4).

Quantitatively, the degree of automation ρ of a production system is proportional to the actual (normalized) capital stock k of the system divided by the capital stock $k_m(y)$ that would be required for *m*aximally automated production of actual output y; in the state of maximally automated production the output elasticity of routine labor would be vanishingly small. The proportionality factor is the degree of capacity utilization of the capital stock, η, defined as the appropriate average over the degrees of capacity utilization of the individual production units that make up the total capital stock. Thus, the degree of automation is given by

$$\rho = \eta \frac{k}{k_m(y)} \quad . \tag{5}$$

Entrepreneurial decisions, aiming at producing a certain quantity of output y within existing technology, determine the absolute magnitude of the total capital stock k, its degree of capacity utilization η, and its degree of automation ρ.

Obviously, ρ and η are functions of (normalized) capital k, labor l, and energy e. They are definitely constrained by $\rho(k, l, e) \leq 1$ and $\eta(k, l, e) \leq 1$, i.e. the maximum degree of automation (at a given time) cannot be exceeded, and a production system cannot operate above design capacity.[8] Here it is important that (productive) energy input into machines and other capital equipment is always limited by their technical design.

However, there is a technical limit to the degree of automation at time t that lies below 1. We call it $\rho_T(t)$. It depends on mass, volume and exergy requirements of the machines, especially information processors, in the capital stock. Imagine the vacuum-tube computers of the 1960s, when the tiny transistor, invented in 1947 by Bardeen, Brattain and Shockley, had not yet diffused into the capital stock. A vacuum-tube computer with the computing power of a 2008 notebook computer would have had a volume of many thousands of cubic meters. In 1960 a degree of automation, that is standard 40 years later in the highly industrialized countries, would have resulted in factories many orders of magnitude bigger than today, probably exceeding the available land area.

In the course of time, the technical limit to automation, $\rho_T(t)$, increases. Its move towards the theoretical limit 1 is facilitated by the density increase of information processors (transitors) on a microchip. According to "Moore's Law" transistor density has doubled every 18 months during the last four decades. It may continue like that for a while, thanks to nano-technological progress. But there is a thermodynamic limit to transistor density, because the electricity required for information processing eventually ends up in heat. If this heat can no longer escape sufficiently rapidly out of the microchip because of too densely packed transistors, it will melt down the conducting elements and destroy the chip. We do not know exactly, how far the technical limit to automation can be pushed. For our purposes, however, it is sufficient to know that at any time t such a limit $\rho_T(t)$ exists.

Since the technical properties of the capital stock do not change with η, the constraint on automation applies to the situation of maximum capacity utilization. With $\eta = 1$ in eq. (5) the (inequality) formulation of an upper limit to automation is: $k/k_m(y) \leq \rho_T(t)$. It is brought into the form of a constraint *equation*, as required by the method of the

[8]Strictly speaking, the limit 1 for η is a sharp technological limit only, when "working at full capacity" means working 24 hours per day and 365 days per year. There are branches of business, where machines have to run less time per day and year in order to be considered as working at full capacity. To keep things simple we disregard these "soft" limits to capacity utilization.

Lagrange multipliers, with the help of the slack variable k_ρ :

$$f_A(K, L, E, t) \equiv \frac{k + k_\rho}{k_m(y)} - \rho_T(t) = 0 \quad .^9 \tag{6}$$

The capital k_ρ has to be added to k so that the total capital stock $k + k_\rho$, working at full capacity, exhausts the technologically possible automation potential $\rho_T(t)$.

Similarly, the formulation of an upper limit to capacity utilization, $\eta(k, l, e) \leq 1$, is brought into the required form of a constraint equation with the help of the slack variables $e_\eta(t)$ and $l_\eta(t)$:

$$f_B(K, L, E, t) \equiv \eta(k, l + l_\eta, e + e_\eta) - 1 = 0 \quad ; \tag{7}$$

$l + l_\eta$ and $e + e_\eta$ are the quantities of labor and energy required for full capacity utilization of the capital stock k at time t. In a rough approximation one may assume that $e_\eta(t)$ and $l_\eta(t)$ are related by $e_\eta(t) = d(t) \cdot l_\eta(t)$. Here, the technological state of the capital stock determines the labor-energy-coupling parameter $d(t)$. The three slack variables k_ρ, e_η and l_η are functions of k, l, e as shown by eqs. (61), (62) and (69) of Appendix C.

We need an explicit functional form for the degree of capacity utilization η. Since η does not change, if k, l and e all change by the same factor, it is a homogeneous function of degree zero: $\eta = \eta(l/k, e/k)$. A trial form can be derived from a Taylor expansion of $\ln \eta[\ln(l/k), \ln(e/k)]$ around some point '0'$\equiv (\ln(l/k)_0, \ln(e/k)_0)$, up to first order in $\ln(l/k) - \ln(l/k)_0$ and $\ln(e/k) - \ln(e/k)_0$. This approximation yields

$$\eta = \eta_0 \left(\frac{l}{k}\right)^\lambda \left(\frac{e}{k}\right)^\nu, \tag{8}$$

where λ and ν are the derivatives of $\ln \eta$ with respect to $\ln l/k$ and $\ln e/k$ in the point '0'. The parameters η_0, λ and ν can be determined (in principle) from empirical data on capacity utilization. Then, combining eq. (7) with eq. (8), one has the complete equation describing constrained capacity utilization.

In order to complete the constraint equation (6) one needs the capital stock $k_m(y)$ required for maximally automated production of a given quantity of output y that is actually produced by k, l and e. Knowledge of the production function $y(k, l, e)$ makes the calculation of $k_m(y)$ possible. The output elasticities in the equilibrium conditions (1) and in equations (35) and (36) for the ratios of Lagrange multipliers are also known, if one knows $y(k, l, e)$. Thus, an appropriate production function is the last remaining element we need in order to have explicit constraint equations and solve the problem of economic equilibrium under technological constraints in principle.

[9] Trivially, any function of k, l, e is also a function of K, L, E via the transformation (3).

IV. LINEX FUNCTION AND SHADOW PRICES

There are two kinds of production functions: 1) production possibility frontiers, that depend on factor inputs required at full capacity, and 2) production functions that depend on the actual factor inputs. For production possibility frontiers there is only one constraint equation, the one on automation. In order to be somewhat more general we consider production functions of the second kind, where both constraint equations apply.

An appropriate function, computed independently from any equilibrium condition, is the Linex production function

$$y = y_L = y_0 e \exp\left[a(2 - \frac{l+e}{k}) + ac(\frac{l}{e} - 1)\right], \tag{9}$$

which depends *lin*early on energy and *ex*ponentially on ratios of capital, labor and energy. It results from the capital-labor-energy-creativity (KLEC) model, which has been developed since 1980. The basic model equations are reproduced in Appendix C. In recent econometric applications by Hall et al. (2001) and Kümmel et al. (2002), the Linex function reproduces economic growth since the 1960s, using primary energy E in the normalized energy variable $e \equiv E/E_0$. Using useful work U instead[10] Ayres and Warr (2005) have reproduced economic growth for 10 decades. In both cases residuals and autocorrelations are small.

The Linex function contains the technology parameters a (capital effectiveness), c (energy requirement of capital), and y_0. These parameters may become time dependent $a(t)$, $c(t)$, and $y_0(t)$. Then, creativity acts in the sense that the Linex function acquires an explicit time dependence: $y = y_L[k, l, e; t]$. (For example, $c(t)$ decreases when investments in energy conservation measures improve the energy efficiency of the capital stock.) The output elasticities of capital, labor, energy, and creativity, defined by $\alpha \equiv (k/y)(\partial y/\partial k)$, $\beta \equiv (l/y)(\partial y/\partial l)$, $\gamma \equiv (e/y)(\partial y/\partial e)$, and $\delta \equiv [(t-t_0)/y](\partial y/\partial t)$, are for the Linex function (9)

$$\alpha = a\frac{l+e}{k}, \quad \beta = al\left(\frac{c}{e} - \frac{1}{k}\right), \quad \gamma = 1 - \alpha - \beta \quad , \tag{10}$$

and

$$\delta = \frac{(t-t_0)}{y_L}\left[\frac{\partial y_L}{\partial a}\frac{\mathrm{d}a}{\mathrm{d}t} + \frac{\partial y_L}{\partial c}\frac{\mathrm{d}c}{\mathrm{d}t} + \frac{\partial y_L}{\partial y_0}\frac{\mathrm{d}y_0}{\mathrm{d}t}\right]. \tag{11}$$

see also Appendix C.

The abovementioned econometric studies yield time-averaged Linex output elasticities that are much smaller for labor and much larger for energy or useful work than the

[10] Useful work U is exergy, multiplied by appropriate conversion efficiencies, and physical work by animals. It has been computed by Ayres et al. (2003).

factor-cost shares. Cointegration analyses, e.g. by Stresing et al. (2008), support these findings.

The capital stock $k_m(y)$ for the maximally automated production of an output y that at time t is produced by the factors $k(t), l(t)$ and $e(t)$ is calculated in Appendix C by equating the actual Linex function (i.e. the one at a given time t) with the Linex function one would have in the state of maximally automated production with unchanged technology parameters. This results in

$$k_m(y) = \frac{e(t)}{c(t)} \exp\left[a(t)c(t)\left(1 + \frac{l(t)}{e(t)}\right) - a(t)\frac{l(t)+e(t)}{k(t)}\right] \quad . \tag{12}$$

With that the constraint equation (6) is complete.

The shadow prices of eq. (2), which translate technological constraints into monetary terms, are as follows for capital, labor and energy:

$$s_K = -\frac{1}{K_0}\left[\mu_1\frac{\partial f_A}{\partial k} + \mu_2\frac{\partial f_B}{\partial k}\right], \quad s_L = -\frac{1}{L_0}\left[\mu_1\frac{\partial f_A}{\partial l} + \mu_2\frac{\partial f_B}{\partial l}\right] \quad , \tag{13}$$

and

$$s_E = -\frac{1}{E_0}\left[\mu_1\frac{\partial f_A}{\partial e} + \mu_2\frac{\partial f_B}{\partial e}\right] \quad . \tag{14}$$

Here the transformtion (3) and the chain rule have been observed.

The ratios of Lagrange multipliers $\mu_1 \equiv \mu_A/\mu$ and $\mu_2 \equiv \mu_B/\mu$ are given by eqs. (35) and (36), where one has to identify $X_1 = K_0 k, X_2 = L_0 l, X_3 = E_0 e$, $\epsilon_1 = \alpha$, $\epsilon_2 = \beta$, $\epsilon_3 = \gamma$, and replace subscripts 1,2,3 by K, L, E. The functions f_A and f_B, which model the technological constraints, are obtained by inserting eq. (12) into eq. (6) and combining eqs. (7) and (8). Appendix C computes the derivatives of f_A and f_B, derives the slack variables, and summarizes the parameters and data required for solving the problem of economic equilibrium under technological constraints for a given economic system.

V. SUMMARY AND DISCUSSION

Maximization of profit or time-integrated utility yields new equilibrium conditions for economic systems whose factors of production are subject to technological constraints. The equilibrium values of the input factors are determined by the conditions that the output elasticity of each factor is equal to the "shadowed" factor cost share. The shadowed cost share is the product of the factor with a two-component price term, divided by the sum over all factors times their respective price terms. Each price term consists of the factor market price plus a shadow price, which is due to the technological constraints. In a model, where capital, labor and energy (exergy) are the factors of production, the constraints affect the degree of automation and the degree of capacity utilization. There are two constraint parameters: the technical limit to the degree of automation and the labor-energy-coupling parameter at full capacity.

The shadow prices depend on the output elasticities. Output elasticities are obtained (independently from equilibrium conditions) as functions of capital, labor, and energy (or useful work). They are solutions of partial differential equations that result from the twice differentiablity requirement on production functions. Technology parameters of the resulting Linex production function have to be determined either by econometric estimations, or, alternatively, by equating the Linex function to the sum of known microeconomic production functions.

We emphasize that the suggested approach to modeling production takes into account explicitly the fact that capital is utilized through energy and labor inputs. The conceptual basis is the notion that output is generated via work performance and information processing through the interaction of the factors capital, labor, and energy (or useful work). Consequently, the utilization rate of capital is endogenous in the model (depending on the ratios of labor to capital and energy to capital), and the relevant output quantity is actual output – not potential output. This differs from alternative interpretations of production functions as production possibility frontiers, which consider potential output (output at full capacity). The latter must employ – and construct for the purpose of numerical model application – utilization-adjusted factor inputs. Such scheme requires additional assumptions that are not needed in the present model, since capacity utilization is endogenous. Production functions that endogenize capacity utilization are evidently the appropriate tool for estimating the economic impact of actual factor inputs.

Another feature of the proposed model, which differs from the concept of production possibility frontiers, is the explicit introduction of that component of technical progress which we call automation, i.e. the substitution of capital and energy for (routine) labor. At a given point in time at given state of technology, automation is limited by the automation potential $\rho_T(t)$. This represents a technological constraint that may be relaxed in the course of technological progress.

If one wants to verify the evolution of past outputs and inputs according to the constrained equilibrium conditions, one needs time series for factor market prices and constraint parameters. Business inquiries and technical analyses of the capital stock may provide the data for the technological constraints. Alternatively, one might assume that in the past the economies have operated in constrained equilibria. Then one can use the empirical time series of inputs and factor market prices in the equilibrium conditions and construct time series for the constraint parameters in the shadow prices.[11]

[11]It is not possible to calculate the s_i directly from eq. (1), if the ϵ_i are known: one obtains quotients whose numerators and denominators vanish because of constant returns to scale.

Extrapolation of these time series into the future, and guesses about the evolution of the technology parameters in the Linex output elasticities, will then allow predictions of economic growth within scenarios including market prices of capital, labor and energy.

Linex output elasticities, which are much smaller for labor and much larger for energy than the cost shares of these factors, have essentially eliminated the Solow residual of neoclassical growth theory. They are consistent with economic equilibrium from profit or utility maximization, if the shadow prices that correspond to constraints from basic engineering relationships between capital, labor and energy are taken into account.[12]

The modified framework for growth theory, consists of a) the equilibrium conditions with shadowed factor cost shares, b) the shadow price equations, and c) macroeconomic production functions that are calculated independently from the equilibrium conditions. It should be helpful for designing growth models that address the problems of climate change and diminishing oil (and gas) reserves. These models may analyze scenarios like the following, and combinations thereof. 1) All energy conservation potentials have been exhausted down to the limits drawn by the laws of thermodynamics, the remaining emissions of greenhouse gases from fossil fuel combustion cannot be sufficiently reduced by carbon capture and sequestration, and the damages from climate change become as severe as likely IPCC scenarios predict. Therefore, governments agree globally on drastic reductions of fossil fuel consumption. 2) World oil production starts to decrease ("Peak Oil"). 3) With sufficiently large investments one develops and installs solar and/or nuclear backstop technologies.[13]

We believe that weighting the options for global warming and peak oil policies will benefit from such analyses within the modified framework for growth theory.

APPENDIX A : PROFIT MAXIMIZATION SUBJECT TO TECHNOLOGICAL CONSTRAINTS

We consider an economic systems with N factors of production $X_1 \dots X_i \dots X_N$, which can vary independently within certain technological constraints. They form the input vector $\vec{X}$. X_1 is capital. Non-routine human contributions to economic progress may be taken into account by time-changing technology parameters of the production function $Y(\vec{X})$.

[12]Output elasticities measure the productive powers of factors. The disequilibrium between productive powers and real cost shares of routine labor and energy explains the pressure to increase automation, substituting productively powerful, cheap energy/capital combinations for productively weak, expensive labor. Globalization is another consequence, because goods and services produced in low-wage countries can be transported cheaply to high-wage countries, thanks to cheap energy and sophisticated logistics.

[13]The spectrum of backstop technologies comprises large-scale photovoltaic or solar thermal power plants in the deserts of earth, solar power satellites in geostationary orbit, small-scale use of renewables wherever possible, and nuclear fusion reactors.

With the help of slack variables the technological constraints on factor combinations, labeled by a, are expressed by equations:

$$f_a(\vec{X}, t) = 0 \quad . \tag{15}$$

The exogenously given prices per unit of factors $X_1 \dots X_i \dots X_N$ are $p_1 \dots p_i \dots p_N$. They form the price vector $\vec{p}(t)$. Thus, total factor cost is

$$FC \equiv \vec{p}(t) \cdot \vec{X}(t) = \sum_{i=1}^{N} p_i(t) X_i(t). \tag{16}$$

The criteria for economic equilibrium under technological constraints are obtained from two different models of optimization by economic agents. The first, more straightforward model treated here, assumes that the decisions of all economic agents result in maximization of profit, which is the difference between the macroeconomic output Y and the total factor cost FC . The second model assumes that society maximizes the time integral of utility of consumption. It is outlined in Appendix B.

The necessary condition for a maximum of output subject to the constraint of fixed cost FC and the technological constraints (15) is: the gradient in factor space of $Y(\vec{X}) + \mu(FC - \vec{p} \cdot \vec{X}) + \sum_a \mu_a f_a(\vec{X}, t)$ must vanish.[14] This yields the equilibrium conditions

$$\frac{\partial Y}{\partial X_i} - \mu \left[p_i - \sum_a \frac{\mu_a}{\mu} \frac{\partial f_a}{\partial X_i} \right] = 0, \quad i = 1 \dots N \quad , \tag{17}$$

where μ and μ_a are Lagrange multipliers.

The equilibrium values of the X_i can be computed from these N equations, if one knows the production function, factor prices and constraints.

Multiplication of eq. (17) with $\frac{X_i}{Y}$ brings the equilibrium conditions into the form

$$\frac{X_i}{Y} \frac{\partial Y}{\partial X_i} = \frac{\mu}{Y} X_i \left[p_i - \sum_a \frac{\mu_a}{\mu} \frac{\partial f_a}{\partial X_i} \right], \qquad i = 1 \dots N \quad , \tag{18}$$

where

$$\frac{X_i}{Y} \frac{\partial Y}{\partial X_i} \equiv \epsilon_i \tag{19}$$

also defines the output elasticity ϵ_i of the production factor X_i.

We assume that the production function $Y(\vec{X})$ is linearly homogeneous in $(X_1 \dots X_i \dots X_N)$, so that we have constant returns to scale:

$$\sum_{i=1}^{N} \epsilon_i = 1 \quad . \tag{20}$$

[14]The sufficient condition for profit maximum involves a sum of second-order derivatives of $Y(\vec{X}) + \mu(FC - \vec{p} \cdot \vec{X}) + \sum_a \mu_a f_a(\vec{X}, t)$. We assume that the extremum of profit at finite X_i *is* the maximum.

Combining the last three equations one obtains

$$Y = Y\sum_{i=1}^{N}\epsilon_i = \mu\sum_{i=1}^{N}X_i\left[p_i - \sum_a \frac{\mu_a}{\mu}\frac{\partial f_a}{\partial X_i}\right] = \mu\left[FC - \sum_{i=1}^{N}X_i\sum_a \frac{\mu_a}{\mu}\frac{\partial f_a}{\partial X_i}\right] \quad , \tag{21}$$

where the total factor cost FC is given by eq. (16). Equation (21) can also be written in the form

$$\mu = \frac{Y}{FC - \sum_{i=1}^{N}X_i\sum_a \frac{\mu_a}{\mu}\frac{\partial f_a}{\partial X_i}} \quad . \tag{22}$$

We insert μ from eq. (22) into eq. (18) and observe eq. (19). This yields the equilibrium conditions in the form

$$\epsilon_i = \frac{X_i\left[p_i - \sum_a \frac{\mu_a}{\mu}\frac{\partial f_a}{\partial X_i}\right]}{FC - \sum_{i=1}^{N}X_i\sum_a \frac{\mu_a}{\mu}\frac{\partial f_a}{\partial X_i}}, \quad i = 1\ldots N \quad . \tag{23}$$

Without technological constraints all μ_a are absent in eq. (23). Then one obtains the non-constrained equilibrium conditions according to which output elasticities $\epsilon_{i,nc}$ are equal to factor cost shares:

$$\epsilon_{i,nc} = \frac{X_i \cdot p_i}{FC} \quad . \tag{24}$$

For the general case we rewrite eq. (23):

$$\epsilon_i = \frac{X_i\left[p_i - \sum_a \frac{\mu_a}{\mu}\frac{\partial f_a}{\partial X_i}\right]}{\sum_{i=1}^{N}X_i\left[p_i - \sum_a \frac{\mu_a}{\mu}\frac{\partial f_a}{\partial X_i}\right]} = \frac{X_i\left[p_i + s_i\right]}{\sum_{i=1}^{N}X_i\left[p_i + s_i\right]} \quad , \tag{25}$$

where s_i is the shadow price of the production factor X_i, defined as

$$s_i \equiv -\sum_a \frac{\mu_a}{\mu}\frac{\partial f_a}{\partial X_i} \quad . \tag{26}$$

The last equality in eq. (25) can also be written as

$$\epsilon_i = \frac{X_i(p_i + s_i)}{(FC + FC_S)}, \quad FC_S \equiv \sum_{i=1}^{N}X_i s_i \quad , \tag{27}$$

where FC_S is the total shadow cost of the input factors.

The change of the standard equilibrium conditions (24), where output elasticities equal factor cost shares, to the modified equilibrium condition (25), where output elasticities equal "shadowed" factor cost shares, complicates the computation of output elasticites: they cannot simply and directly be obtained from the equilibrium conditions, because the shadow prices depend on the output elasticities. This is shown in the following for the case of a three-factor model, $N = 3$, with two constraint equations

$$f_A(X_1(t), X_2(t), X_3(t), t) = 0, \quad f_B(X_1(t), X_2(t), X_3(t), t) = 0. \tag{28}$$

In this case the sum over subscripts a in eq. (26) contains only two terms A and B. It is convenient to abbreviate

$$\mu_1 \equiv \frac{\mu_A}{\mu}, \quad \mu_2 \equiv \frac{\mu_B}{\mu}, \tag{29}$$

and define the partial derivatives of the two constraint equations as

$$f_{Ai} \equiv \frac{\partial f_A}{\partial X_i}, \quad f_{Bi} \equiv \frac{\partial f_B}{\partial X_i}, \quad i = 1, 2, 3 \quad . \tag{30}$$

With that the equilibrum conditions (23) become

$$\epsilon_i = \frac{X_i \left[p_i - \mu_1 f_{Ai} - \mu_2 f_{Bi}\right]}{FC - \sum_{i=1}^{3} X_i \left[\mu_1 f_{Ai} + \mu_2 f_{Bi}\right]}, \quad i = 1, 2, 3 \quad . \tag{31}$$

If one resolves the two independent ratios

$$\frac{\epsilon_1}{\epsilon_2} = \frac{X_1 \left[p_1 - \mu_1 f_{A1} - \mu_2 f_{B1}\right]}{X_2 \left[p_2 - \mu_1 f_{A2} - \mu_2 f_{B2}\right]} \tag{32}$$

and

$$\frac{\epsilon_1}{\epsilon_3} = \frac{X_1 \left[p_1 - \mu_1 f_{A1} - \mu_2 f_{B1}\right]}{X_3 \left[p_3 - \mu_1 f_{A3} - \mu_2 f_{B3}\right]}, \tag{33}$$

with respect to μ_1 and μ_2, using the definitions

$$R_{21} \equiv \frac{X_2 \epsilon_1}{X_1 \epsilon_2}, \quad R_{31} \equiv \frac{X_3 \epsilon_1}{X_1 \epsilon_3} \quad , \tag{34}$$

and performing some algebraic manipulations, one obtains

$$\mu_1 = \frac{(p_1 - p_2 R_{21})}{f_{A1} - f_{A2} R_{21}} + \frac{f_{B2} R_{21} - f_{B1}}{f_{A1} - f_{A2} R_{21}} \cdot \mu_2 \quad , \tag{35}$$

and

$$\mu_2 = \frac{(p_1 - p_3 R_{31})(f_{A1} - f_{A2} R_{21}) - (p_1 - p_2 R_{21})(f_{A1} - f_{A3} R_{31})}{(f_{B2} R_{21} - f_{B1})(f_{A1} - f_{A3} R_{31}) - (f_{B3} R_{31} - f_{B1})(f_{A1} - f_{A2} R_{21})} \quad . \tag{36}$$

Thus, the Lagrange parameter ratios $\mu_1 \equiv \mu_A/\mu$ and $\mu_2 \equiv \mu_B/\mu$ are elasticity dependent via the quotients (34), and so are the shadow prices s_i, where $i = 1, 2, 3$ and $a = A, B$ in eq. (26).

APPENDIX B: INTERTEMPORAL UTILITY OPTIMIZATION SUBJECT TO CONSTRAINTS

In their paper "A complete capital model involving heterogeneous capital goods", Samuelson and Solow (1956) review Ramsey's (1928) problem via their equation (1), on which they comment: "In words, society maximizes the (undiscounted) integral of all future utilities of consumption subject to the fact that the sum of current consumption and of current capital formation is limited by what the current capital stock can produce." Extending Ramsey's one-capital-good theory to their many-goods model they continue with maximizing undiscounted integrals of utility. One could argue the disregard of discounting.[15] On the other hand, Ramsey (1928), Arrow (1973), Solow (1974)[16], and others question time preferences and discounting because of ethical reasons. Stern (2007, 2008), following them, reasons that the only sound ethical basis for placing less value on the utility of future generations is the uncertainty whether or not the world will exist, or whether those generations will all be present. He uses a pure time discount rate of 0.1%. (This corrresponds to a probability of 90.5% that humanity will not yet have perished within 100 years.) Stern (2008) defends his low discount rate against criticism from a number of authors. Given the diverging views on discounting, and for the sake of simplicity, we follow the optimization procedure in Section I of Samuelson and Solow's paper with the following modifications. 1. There is not one variable factor of production but N different factors, which we label $X_1, \ldots X_i, \ldots X_N$. 2. There are contraints on magnitudes and combinations of these factors. 3. As in Hellwig et al. (2000), utility optimization is done within finite time horizons. Thus, on our second way of deriving equilibrium conditions we assume that society maximizes the (undiscounted) integral W of utility U of consumption C between the times t_0 and t_1,[17,18] where the sum of consumption and capital formation at any time t between t_0 and t_1 is limited by what the capital stock $X_1(t)$ in combination with other production factors $X_i(t)$ can produce with due regard of the contraints on these factors. Thus, we have the optimization problem: Maximize

$$W[s] = \int_{t_0}^{t_1} U[C]dt, \tag{37}$$

subject to constraints.

$W[s]$ is a functional of the curve $[s]$ along which the production factors evolve. This curve depends on the variables that enter C. In general, utility may depend on many variables. In the present case the utility function $U[C]$ depends on output minus capital formation.

[15] In fact, Hellwig et al. (2000) propose a scheme of intertemporal utility optimization subject to the constraint that "maintenance of capital is ... a minimum standard which has to be placed above utility considerations." Discounting is taken into account, but at imperfect capital markets discount rates are ex ante unknown. They are determined by the normalized utility gradient evaluated at the optimum within context-dependent preferences. They may not coincide with the discount rates used for the definition of capital.

[16] Solow (1974, p.9) says "we ought to act as if the social rate of time preference were zero (though we would simultaneously discount future consumption if we expected the future to be richer than the present)".

[17] Modern notation for the integral to be maximized is W. Samuelson and Solow call it J. We define output Y as compared to their notation $f(S)$. Their term S seems to be what they call "abstract capital substance". The rate of change dS/dt must be interpreted as the sum of investment in new capital formation plus replacement of depreciated capital

[18] If one multiplied $U[C]$ by $\exp(-\delta t)$ with the pure time discount rate δ in the integrand of eq. (37), one would have to subtract $\delta \frac{dU}{dC}$ from $\frac{d}{dt}(\frac{dU}{dC})$ in eqs. (50) and (51).

Output (per unit time) is described by the macroeconomic production function $Y(\vec{X})$, where $\vec{X} \equiv (X_1, \dots X_i, \dots X_N)$. Part of Y goes into consumption C and the rest into new capital formation $\dot{X}_1 \equiv \frac{dX_1}{dt}$ plus replacement of depreciated capital. As usual we approximate the annual replacement rate by $\delta^d X_1$, where δ^d is the depreciation rate.[19] Then consumption (per unit time) is $C = Y(\vec{X}) - \dot{X}_1 - \delta^d X_1$. For the optimization of W we need the vector of the prices p_i per unit of factor X_i. This is $\vec{p} \equiv (p_1 \dots p_i \dots p_N)$. Economic research institutions provide the price of capital utilization p_1 as the sum of net interest, depreciation and state influences. Furtheron we use this price. Since it already includes depreciation, we can omit explicit reference to the depreciation rate,[20] so that consumption is given by

$$C = Y(\vec{X}) - \dot{X}_1 \quad . \tag{38}$$

The magnitude of the factors is constrained in the maximization of W by the requirement that total factor cost

$$\vec{p}(t) \cdot \vec{X}(t) = \sum_{i=1}^{N} p_i(t) X_i(t) \tag{39}$$

has finite magnitudes $FC(t)$.

Let there be other, technological constraints on the facor inputs X_i that limit the technically accessible factor space. Indicating them by the label a their equations are written with the help of slack variables as

$$f_a(\vec{X}, t) = 0. \tag{40}$$

Then, with the (generally time dependent) Lagrange multipliers μ, μ_a, the optimization problem becomes:
Maximize

$$W[s] = \int_{t_0}^{t_1} dt \left\{ U[C(\vec{X}, \dot{X}_1)] + \mu(FC(t) - \vec{p} \cdot \vec{X}) + \sum_a \mu_a f_a(\vec{X}, t) \right\} \quad . \tag{41}$$

$W[s]$ is a functional of the curve $[s] = \{t, \vec{X} : \vec{X} = \vec{X}(t), \quad t_0 \leq t \leq t_1\}$. Consider another curve $[s, \vec{h}] = \{t, \vec{X} : \vec{X} = \vec{X}(t) + \vec{h}(t), \quad t_0 \leq t \leq t_1\}$ close to $[s]$, which goes through the same end points so that $\vec{h}(t_1) = 0$ and $\vec{h}(t_0) = 0$. Its functional is

$$W[s, \vec{h}] = \int_{t_0}^{t_1} dt \left\{ U[C(\vec{X} + \vec{h}, \dot{X}_1 + \dot{h}_1)] + \mu \left(FC(t) - \vec{p} \cdot (\vec{X} + \vec{h}) \right) + \sum_a \mu_a f_a(\vec{X} + \vec{h}, t) \right\} \quad . \tag{42}$$

[19]The year is the natural time unit, because the annual cycle of seasons is decisive for agriculture, and important for construction. It also structures education, vacations (hence tourism and transportation) and some other industrial activities in the moderate climate zones. Thus, for practical purposes $Y(\vec{X})$ and $\dot{X}_1 + \delta^d X_1$ are annual output and annual capital formation, respectively.

[20]If we keept $\delta^d X_1$ in the following optimization procedure, we would have a term proportional to $\delta^d X_1$ added to p_1 everywhere.

Since $\vec{h}$ is small, the integrand can be approximated by its Taylor expansion up to first order in $\vec{h}$ and $\dot{h}_1$. The necessary condition for a maximum of W is that the variation of W with respect to $\vec{h}$ vanishes:

$$\delta W \equiv W[s,\vec{h}] - W[s] = \int_{t_0}^{t_1} dt \left\{ \delta U - \mu \vec{p}\cdot\vec{h} + \sum_a \mu_a \sum_{i=1}^{N} \frac{\partial f_a}{\partial X_i} h_i \right\} = 0 \quad . \tag{43}$$

With the chain rule one obtains

$$\delta U \equiv U[C(\vec{X}+\vec{h}, \dot{X}_1 + \dot{h}_1)] - U[C(\vec{X},\dot{X}_1)] = \frac{dU}{dC} dC = \frac{dU}{dC}\left[\sum_{i=1}^{N} \frac{\partial C}{\partial X_i} h_i + \frac{\partial C}{\partial \dot{X}_1}\dot{h}_1\right] \quad . \tag{44}$$

Partial integration yields

$$\begin{aligned} \int_{t_0}^{t_1} dt \frac{dU}{dC}\frac{\partial C}{\partial \dot{X}_1}\dot{h}_1 &= \left[\frac{dU}{dC}\frac{\partial C}{\partial \dot{X}_1} h_1(t)\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} dt \frac{d}{dt}\left(\frac{dU}{dC}\frac{\partial C}{\partial \dot{X}_1}\right) h_1(t) \\ &= -\int_{t_0}^{t_1} dt \frac{d}{dt}\left(\frac{dU}{dC}\frac{\partial C}{\partial \dot{X}_1}\right) h_1(t), \end{aligned} \tag{45}$$

because $h_1(t_1) = 0 = h_1(t_0)$. Combination of eqs. (43) - (45) results in

$$\begin{aligned} \delta W &= \int_{t_0}^{t_1} dt \left\{ \frac{dU}{dC}\sum_{i=1}^{N}\frac{\partial C}{\partial X_i} h_i - \frac{d}{dt}\left(\frac{dU}{dC}\frac{\partial C}{\partial \dot{X}_1}\right) h_1(t) - \mu \sum_{i=1}^{N} p_i h_i + \sum_a \mu_a \sum_{i=1}^{N} \frac{\partial f_a}{\partial X_i} h_i \right\} \\ &= 0 \quad . \end{aligned} \tag{46}$$

Since the small h_i are arbitrary for $t_0 < t < t_1$, the integral can only vanish, if the coefficients of the h_i vanish in the integrand. This yields the following conditions for $\delta W = 0$, i.e. for equilibrium of the economic system:

$$\begin{aligned} \frac{dU}{dC}\frac{\partial C}{\partial X_1} - \frac{d}{dt}\left(\frac{dU}{dC}\frac{\partial C}{\partial \dot{X}_1}\right) - \mu p_1 + \sum_a \mu_a \frac{\partial f_a}{\partial X_1} &= 0 \\ \frac{dU}{dC}\frac{\partial C}{\partial X_i} - \mu p_i + \sum_a \mu_a \frac{\partial f_a}{\partial X_i} &= 0, \quad i = 2 \ldots N. \end{aligned} \tag{47}$$

(Identifying $U[C(\vec{X},\dot{X}_1)]$ with the Lagrangian $L(\vec{X},\dot{X}_1)$, one notes the formal equivalence of these equations with constrained Lagrange equations of motion in classical mechanics.)

With C from eq. (38) these N equilibrium conditions turn into

$$\frac{dU}{dC}\frac{\partial Y}{\partial X_1} - \mu p_1 + \sum_a \mu_a \frac{\partial f_a}{\partial X_1} = -\frac{d}{dt}\left(\frac{dU}{dC}\right), \tag{48}$$

$$\frac{dU}{dC}\frac{\partial Y}{\partial X_i} - \mu p_i + \sum_a \mu_a \frac{\partial f_a}{\partial X_i} = 0 \quad , \quad i = 2 \ldots N \quad . \tag{49}$$

Equations (48), (49) are the general equilibrium conditions for an economic system subject to cost limits and technological constraints,[21] if the behavioral assumption is that society optimizes time-integrated utility.

We evaluate eqs. (48) and (49) along the lines that lead from eq. (17) to eqs. (22) and (23) in Appendix A. Using the Kronecker delta $\delta_{i,1}$, which is 1 for $i = 1$ and 0 otherwise, we obtain

$$\mu = \frac{Y\frac{dU}{dC}}{FC - \sum_{i=1}^{N} X_i \left[\sum_a \frac{\mu_a}{\mu}\frac{\partial f_a}{\partial X_i} + \delta_{i,1}\frac{1}{\mu}\frac{d}{dt}(\frac{dU}{dC})\right]} \tag{50}$$

and

$$\epsilon_i = \frac{X_i\left[p_i - \sum_a \frac{\mu_a}{\mu}\frac{\partial f_a}{\partial X_i} - \delta_{i,1}\frac{1}{\mu}\frac{d}{dt}(\frac{dU}{dC})\right]}{FC - \sum_{i=1}^{N} X_i \left[\sum_a \frac{\mu_a}{\mu}\frac{\partial f_a}{\partial X_i} + \delta_{i,1}\frac{1}{\mu}\frac{d}{dt}(\frac{dU}{dC})\right]}, \quad i = 1\ldots N \quad . \tag{51}$$

The N equilibrium conditions represented by eq. (51) agree with the equilibrium conditions (23) of Appendix A except for the term $\delta_{i,1}\frac{1}{\mu}\frac{d}{dt}(\frac{dU}{dC})$. This term originates from eqs. (38) and (45) and is due to taking capital formation into account in intertemporal utility optimization, whereas capital formation is no issue in profit optimization. This term modifies the shadow prices correspondingly, especially that of capital. If $\frac{d}{dt}\left(\frac{dU}{dC}\right)$ vanishes, eq. (51) turns into eq. (23).

Suppose one can disregard decreasing marginal utility and approximate the utility function $U(C)$ by a linear function in C. For instance, if the function of decreasing marginal utility is $U(C) = C_0 \ln\frac{C}{C_0} + U_0$, and if it can be approximated by its Taylor expansion up to first order in $\frac{C}{C_0} - 1$, one has $U(C) \approx C - C_0 + U_0$.[22] Then $\frac{d}{dt}\left(\frac{dU}{dC}\right) = 0$, and both optimization models yield the same shadow prices.

APPENDIX C: KLEC MODEL AND EXPLICIT CONSTRAINT EQUATIONS

The Linex function (9) is an integral of the "growth equation"[23]

$$\frac{\mathrm{d}y}{y} = \alpha\cdot\frac{\mathrm{d}k}{k} + \beta\cdot\frac{\mathrm{d}l}{l} + \gamma\cdot\frac{\mathrm{d}e}{e} + \delta\cdot\frac{\mathrm{d}t}{t - t_0} \tag{52}$$

at fixed time t between the points $(y_0, 1, 1, 1)$ and (y_L, k, l, e) of four-dimensional output-factor space. In this integral the output elasticities, defined by $\alpha \equiv (k/y)(\partial y/\partial k)$, $\beta \equiv (l/y)(\partial y/\partial l)$, and $\gamma \equiv (e/y)(\partial y/\partial e)$, are the special solutions

$$\alpha = a(l+e)/k, \quad \beta = al\left(c/e - 1/k\right), \quad \gamma = 1 - \alpha - \beta \tag{53}$$

[21]For $\mu = 0 = \mu_a$ eqs. (48) and (49) correspond to eq. (2) of Samuelson and Solow (1956). If y does not depend explicitly on time t, they imply the conservation law $U + \frac{dU}{dC}\dot{X}_1 = \text{constant}$. The conserved Legendre transform of utility, $U + \frac{dU}{dC}\dot{X}_1$, corresponds to the Hamiltonian in classical mechanics.

[22]A linear approximation of $\ln x$ is acceptable for $x < 4$.

[23]The growth equation results from the total differential for any linearly homogeneous production function $y[k, l, e; t]$

of the partial differential equations

$$k\frac{\partial\alpha}{\partial k} + l\frac{\partial\alpha}{\partial l} + e\frac{\partial\alpha}{\partial e} = 0, \qquad k\frac{\partial\beta}{\partial k} + l\frac{\partial\beta}{\partial l} + e\frac{\partial\beta}{\partial e} = 0, \qquad l\frac{\partial\alpha}{\partial l} = k\frac{\partial\beta}{\partial k}, \tag{54}$$

which result from the twice differentiability requirement on production functions, observing constant returns to scale.[24] Equation (11) defines δ. The most general solutions of these equations are:

$$\alpha = A\left(\frac{l}{k}, \frac{e}{k}\right), \qquad \beta = \int \frac{l}{k}\frac{\partial A}{\partial l}\mathrm{d}k + J\left(\frac{l}{e}\right), \tag{55}$$

where A and J are any differentiable functions of their arguments.

The Linex output elasticities satisfy the asymptotic boundary conditions $\alpha \to 0$, if $(l+e)/k \to 0$, and $\beta \to 0$, if $k \to k_m(y)$ and $e \to ck_m(y)$. Here, $e_m \equiv ck_m(y)$ is the energy input into the maximally automated capital stock $k_m(y)$ working at full capacity. The asymptotic boundary condition for α incorporates the law of diminishing returns: machines don't run without energy and (still) require people for handling them; thus, labor and energy are complementary to capital; if their ratio to capital decreases, the output of an additional unit of capital decreases, too. The asymptotic boundary condition for β describes the effect of energy and capital substituting for labor. When estimating the Linex function the constraints $\alpha \geq 0$, $\beta \geq 0$, and $\gamma = 1 - \alpha - \beta \geq 0$ must be observed, because it is assumed that entrepreneurs will avoid a state of the economy where the increase of an input will result in a decrese of output. These mathematical constraints fix, e.g., lower and upper limits to e.

The capital stock $k_m(y)$ for maximally automated production of output y at time t, required in constraint equation (6), can be calculated from the Linex production function y_L of eq. (9) with an explicit time dependence via time-dependent technology parameters $a(t)$ and $c(t)$ by demanding that

$$y_L[k, l, e; t] = y_L[k_m, l_m, e_m = ck_m; t]. \tag{56}$$

The routine labor l_m that remains in the state of maximum automation is certainly much smaller than k_m. If one neglects $l_m/k_m \ll 1$, eq. (56) becomes

$$y_0 e \exp\left[a(t)(2 - \frac{l+e}{k}) + a(t)c(t)(\frac{l}{e} - 1)\right] = y_0 c(t) k_m(y) \exp\left[a(t)(2 - c(t)) - a(t)c(t)\right]. \tag{57}$$

This yields equation (12), i.e. the capital stock for the maximally automated production of an output y that at time t is produced by the factors $k(t), l(t)$ and $e(t)$:

$$k_m(y) = \frac{e(t)}{c(t)} \exp\left[a(t)c(t)\left(1 + \frac{l(t)}{e(t)}\right) - a(t)\frac{l(t) + e(t)}{k(t)}\right] \quad . \tag{58}$$

[24]The trivial solutions of eqs. (54) are constants. If one inserts them into eq. (52) and integrates at fixed t, one obtains the energy-dependent Cobb-Douglas function.

Inserting $k_m(y)$ into eq. (6), with the technical limit to automation $\rho_T(t)$ and the slack variable k_ρ, one obtains the explicit equation for the technological constraint on automation as

$$f_A(K,L,E,t) \equiv \frac{(k+k_\rho)}{k_m(y)} - \rho_T(t) = (k+k_\rho)\frac{c}{e}\exp\left[-ac(1+\frac{l}{e}) + a\frac{l+e}{k}\right] - \rho_T(t) = 0 \quad . \tag{59}$$

Here, and in the following, we drop the time arguments of factors and parameters for the sake of simplicity.

The equation for the constraint on capacity utilization results from eqs. (7) and (8) as

$$f_B(K,L,E,t) \equiv \eta_0 \left(\frac{l+l_\eta(t)}{k}\right)^\lambda \left(\frac{e+e_\eta(t)}{k}\right)^\nu - 1 = 0 \quad . \tag{60}$$

Eqs. (59) and (60) yield the slack-variable relations

$$k + k_\rho = k_m(y)\rho_T(t) \quad , \tag{61}$$

$$e + e_\eta = \frac{k}{\eta_0^{1/\nu}\left(\frac{l+l_\eta}{k}\right)^{\lambda/\nu}} \quad . \tag{62}$$

The derivatives of f_A and f_B are calculated by observing eqs. (3) and the chain rule so that $\partial f_A/\partial K = (1/K_0)(\partial f_A/\partial k)$ etc. From eqs. (59)-(62) we obtain

$$\frac{\partial f_A}{\partial k} = \frac{1}{k_m(y)} - a\frac{l+e}{k^2}\rho_T \tag{63}$$

$$\frac{\partial f_B}{\partial k} = -\frac{\lambda+\nu}{k} \tag{64}$$

$$\frac{\partial f_A}{\partial l} = -a\left(\frac{c}{e} - \frac{1}{k}\right)\rho_T \tag{65}$$

$$\frac{\partial f_B}{\partial l} = \frac{\lambda}{l+l_\eta} \tag{66}$$

$$\frac{\partial f_A}{\partial e} = \left(\frac{a}{k} + \frac{acl}{e^2} - \frac{1}{e}\right)\rho_T \tag{67}$$

$$\frac{\partial f_B}{\partial e} = \frac{\nu}{e+e_\eta} = \frac{\nu}{k}\eta_0^{1/\nu}\left(\frac{l+l_\eta}{k}\right)^{\lambda/\nu} . \tag{68}$$

Inserting them into eqs. (13) and (14) one has the explicit equations for the shadow prices.

In order to compute the shadow prices from the general theoretical framework for an existing economic system one has to take the following steps. 1) The technology parameters a and c have to be determined econometrically for the system.[25] 2) In a

[25] Examples for Germany, Japan and the USA are given by Kümmel et al. (2002) and Ayres and Warr (2005).

rough approximation one may assume proportionality between the slack variables in the constraint on capacity utilization:

$$e_\eta(t) = d(t) \cdot l_\eta(t); \tag{69}$$

here $d(t)$ is the second constraint parameter besides $\rho_T(t)$. We call it the "labor-energy-coupling parameter at full capacity". Ideally, one should be able to determine it from measurements of the energy and labor increases required in order to go from any degree of capacity utilization to 1. Then l_η can be calculated from eq. (62) as a function of k, l, e. With that eq. (69) yields also e_η as a function of k, l, e. 3) The multiplier η_0 and the exponents λ and ν may be obtained by fitting the phenomenological η of eq. (8) to empirical time series of η, which are available from economic research institutions. 4) The technical limit $\rho_T(t)$ to the degree of automation can be any number between 0 and 1. General business inquiries should give clues to it. Alternatively, one has to compute the time series of the shadow prices (13) and (14) for a number of scenarios for $\rho_T(t)$.

Inserting the parameters obtained from these steps into the equations that yield the shadow prices (13) and (14), and combining the shadow prices with the deflated time series of factor prices p_K, p_L, and p_E in eq. (1) for $N = 3$, solves the problem of economic equilibrium under technological constraints within the KLEC model.

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