This book applies the deductive method to programming by affiliating programs with the abstract mathematical theories that enable them to work. Specification of these theories, algorithms written in terms of these theories, and theorems and lemmas describing their properties are presented together. The implementation of the algorithms in a real programming language is central to the book. While the specifications, which are addressed to human beings, should, and even must, combine rigor with appropriate informality, the code, which is addressed to the computer, must be absolutely precise even while being general.

As with other areas of science and engineering, the appropriate foundation of programming is the deductive method. It facilitates the decomposition of complex systems into components with mathematically specified behavior. That, in turn, is a necessary precondition for designing efficient, reliable, secure, and economical software.

The book is addressed to those who want a deeper understanding of programming, whether they are full-time software developers, or scientists and engineers for whom programming is an important part of their professional activity.

The book is intended to be read from beginning to end. Only by reading the code, proving the lemmas, and doing the exercises can readers gain understanding of the material. In addition, we suggest several projects, some open-ended. While the book is terse, a careful reader will eventually see the connections between its parts and the reasons for our choice of material. Discovering the architectural principles of the book should be the reader’s goal.

We assume an ability to do elementary algebraic manipulations.\textsuperscript{1} We also assume familiarity with the basic vocabulary of logic and set theory at the level of undergraduate courses on discrete mathematics; Appendix A summarizes the notation that we use. We provide definitions of a few concepts of abstract algebra when they are needed.

\textsuperscript{1} For a refresher on elementary algebra, we recommend Chrystal [1904].
needed to specify algorithms. We assume programming maturity and understanding of computer architecture\(^2\) and fundamental algorithms and data structures.\(^3\)

We chose C++ because it combines powerful abstraction facilities with faithful representation of the underlying machine.\(^4\) We use a small subset of the language and write requirements as structured comments. We hope that readers not already familiar with C++ are able to follow the book. Appendix B specifies the subset of the language used in the book.\(^5\) Wherever there is a difference between mathematical notation and C++, the typesetting and the context determine whether the mathematical or C++ meaning applies. While many concepts and programs in the book have parallels in STL (the C++ Standard Template Library), the book departs from some of the STL design decisions. The book also ignores issues that a real library, such as STL, has to address: namespaces, visibility, inline directives, and so on.

Chapter 1 describes values, objects, types, procedures, and concepts. Chapters 2–5 describe algorithms on algebraic structures, such as semigroups and totally ordered sets. Chapters 6–11 describe algorithms on abstractions of memory. Chapter 12 describes objects containing other objects. The Afterword presents our reflections on the approach presented by the book.

Acknowledgments

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2. We recommend Patterson and Hennessy [2007].
3. For a selective but incisive introduction to algorithms and data structures, we recommend Tarjan [1983].
4. The standard reference is Stroustrup [2000].
5. The code in the book compiles and runs under Microsoft Visual C++ 9 and g++ 4. This code, together with a few trivial macros that enable it to compile, as well as unit tests, can be downloaded from www.elementsofprogramming.com.
The book has benefited significantly from the contributions of our editor, Peter Gordon, our project editor, Elizabeth Ryan, our copy editor, Evelyn Pyle, and the editorial reviewers: Matt Austern, Andrew Koenig, David Musser, Arch Robison, Jerry Schwarz, Jeremy Siek, and John Wilkinson.

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Finally, we are grateful to all the people who taught us through their writings or in person, and to the institutions that allowed us to deepen our understanding of programming.
Chapter 2
Transformations and Their Orbits

This chapter defines a transformation as a unary regular function from a type to itself. Successive applications of a transformation starting from an initial value determine an orbit of this value. Depending only on the regularity of the transformation and the finiteness of the orbit, we implement an algorithm for determining orbit structures that can be used in different domains. For example, it could be used to detect a cycle in a linked list or to analyze a pseudorandom number generator. We derive an interface to the algorithm as a set of related procedures and definitions for their arguments and results. This analysis of an orbit-structure algorithm allows us to introduce our approach to programming in the simplest possible setting.

2.1 Transformations

While there are functions from any sequence of types to any type, particular classes of signatures commonly occur. In this book we frequently use two such classes: homogeneous predicates and operations. Homogeneous predicates are of the form $T \times \cdots \times T \to \text{bool}$; operations are functions of the form $T \times \cdots \times T \to T$. While there are n-ary predicates and n-ary operations, we encounter mostly unary and binary homogeneous predicates and unary and binary operations.

A predicate is a functional procedure returning a truth value:

\[
\text{Predicate}(P) \triangleq \text{FunctionalProcedure}(P) \land \text{Codomain}(P) = \text{bool}
\]
A homogeneous predicate is one that is also a homogeneous function:

\[
\text{HomogeneousPredicate}(P) \triangleq \text{Predicate}(P) \land \text{HomogeneousFunction}(P)
\]

A unary predicate is a predicate taking one parameter:

\[
\text{UnaryPredicate}(P) \triangleq \text{Predicate}(P) \land \text{UnaryFunction}(P)
\]

An operation is a homogeneous function whose codomain is equal to its domain:

\[
\text{Operation}(\text{Op}) \triangleq \text{HomogeneousFunction}(\text{Op}) \land \text{Codomain}(\text{Op}) = \text{Domain}(\text{Op})
\]

Examples of operations:

```c
int abs(int x) {
    if (x < 0) return -x; else return x;
} // unary operation
```

```c
double euclidean_norm(double x, double y) {
    return sqrt(x * x + y * y);
} // binary operation
```

```c
double euclidean_norm(double x, double y, double z) {
    return sqrt(x * x + y * y + z * z);
} // ternary operation
```

**Lemma 2.1** \(\text{euclidean\_norm}(x, y, z) = \text{euclidean\_norm}(\text{euclidean\_norm}(x, y), z)\)

This lemma shows that the ternary version can be obtained from the binary version. For reasons of efficiency, expressiveness, and, possibly, accuracy, the ternary version is part of the computational basis for programs dealing with three-dimensional space.
A procedure is *partial* if its definition space is a subset of the direct product of the types of its inputs; it is *total* if its definition space is equal to the direct product. We follow standard mathematical usage, where partial function includes total function. We call partial procedures that are not total *nontotal*. Implementations of some total functions are nontotal on the computer because of the finiteness of the representation. For example, addition on signed 32-bit integers is nontotal.

A nontotal procedure is accompanied by a precondition specifying its definition space. To verify the correctness of a call of that procedure, we must determine that the arguments satisfy the precondition. Sometimes, a partial procedure is passed as a parameter to an algorithm that needs to determine at runtime the definition space of the procedural parameter. To deal with such cases, we define a *definition-space predicate* with the same inputs as the procedure; the predicate returns true if and only if the inputs are within the definition space of the procedure. Before a nontotal procedure is called, either its precondition must be satisfied, or the call must be guarded by a call of its definition-space predicate.

**Exercise 2.1** Implement a definition-space predicate for addition on 32-bit signed integers.

This chapter deals with unary operations, which we call *transformations*:

\[
\text{Transformation}(F) \triangleq \\
\text{Operation}(F) \\
\land \text{UnaryFunction}(F) \\
\land \text{DistanceType} : \text{Transformation} \rightarrow \text{Integer}
\]

We discuss DistanceType in the next section.

Transformations are self-composable: \( f(x), f(f(x)), f(f(f(x))) \), and so on. The definition space of \( f(f(x)) \) is the intersection of the definition space and result space of \( f \). This ability to self-compose, together with the ability to test for equality, allows us to define interesting algorithms.

When \( f \) is a transformation, we define its powers as follows:

\[
f^n(x) = \begin{cases} x & \text{if } n = 0, \\ f^{n-1}(f(x)) & \text{if } n > 0 \end{cases}
\]
To implement an algorithm to compute $f^n(x)$, we need to specify the requirement for an integer type. We study various concepts describing integers in Chapter 5. For now we rely on the intuitive understanding of integers. Their models include signed and unsigned integral types, as well as arbitrary-precision integers, with these operations and literals:

<table>
<thead>
<tr>
<th></th>
<th>Specifications</th>
<th>C++</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sum</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Difference</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>Product</td>
<td>.</td>
<td>*</td>
</tr>
<tr>
<td>Quotient</td>
<td>/</td>
<td>/</td>
</tr>
<tr>
<td>Remainder</td>
<td>mod</td>
<td>%</td>
</tr>
<tr>
<td>Zero</td>
<td>0</td>
<td>I(0)</td>
</tr>
<tr>
<td>One</td>
<td>1</td>
<td>I(1)</td>
</tr>
<tr>
<td>Two</td>
<td>2</td>
<td>I(2)</td>
</tr>
</tbody>
</table>

where I is an integer type.

That leads to the following algorithm:

```cpp
template<typename F, typename N>
requires(Transformation(F) && Integer(N))
Domain(F) power_unary(Domain(F) x, N n, F f)
{
    // Precondition: $n \geq 0 \land (\forall i \in N) 0 < i \leq n \Rightarrow f^n(x)$ is defined
    while (n != N(0)) {
        n = n - N(1);
        x = f(x);
    }
    return x;
}
```

### 2.2 Orbits

To understand the global behavior of a transformation, we examine the structure of its *orbits*: elements reachable from a starting element by repeated applications of the transformation. $y$ is *reachable* from $x$ under a transformation $f$ if for some $n \geq 0$, $y = f^n(x)$. $x$ is *cyclic* under $f$ if for some $n \geq 1$, $x = f^n(x)$. $x$ is *terminal* under $f$ if and only if $x$ is not in the definition space of $f$. The *orbit* of $x$ under a transformation $f$ is the set of all elements reachable from $x$ under $f$.  

Lemma 2.2  An orbit does not contain both a cyclic and a terminal element.

Lemma 2.3  An orbit contains at most one terminal element.

If \( y \) is reachable from \( x \) under \( f \), the distance from \( x \) to \( y \) is the least number of transformation steps from \( x \) to \( y \). Obviously, distance is not always defined.

Given a transformation type \( F \), DistanceType\((F)\) is an integer type large enough to encode the maximum number of steps by any transformation \( f \in F \) from one element of \( T = \text{Domain}(F) \) to another. If type \( T \) occupies \( k \) bits, there can be as many as \( 2^k \) values but only \( 2^k - 1 \) steps between distinct values. Thus if \( T \) is a fixed-size type, an integral type of the same size is a valid distance type for any transformation on \( T \). (Instead of using the distance type, we allow the use of any integer type in power_unary, since the extra generality does not appear to hurt there.) It is often the case that all transformation types over a domain have the same distance type. In this case the type function DistanceType is defined for the domain type and defines the corresponding type function for the transformation types.

The existence of DistanceType leads to the following procedure:

```cpp
template<typename F>
  requires(Transformation(F))
DistanceType(F) distance(Domain(F) x, Domain(F) y, F f)
{
  // Precondition: y is reachable from x under f
  typedef DistanceType(F) N;
  N n(0);
  while (x != y) {
    x = f(x);
    n = n + N(1);
  }
  return n;
}
```

Orbits have different shapes. An orbit of \( x \) under a transformation is

- **infinite** if it has no cyclic or terminal elements
- **terminating** if it has a terminal element
- **circular** if \( x \) is cyclic
- **\( \rho \)-shaped** if \( x \) is not cyclic, but its orbit contains a cyclic element

An orbit of \( x \) is **finite** if it is not infinite. Figure 2.1 illustrates the various cases.
The orbit cycle is the set of cyclic elements in the orbit and is empty for infinite and terminating orbits. The orbit handle, the complement of the orbit cycle with respect to the orbit, is empty for a circular orbit. The connection point is the first cyclic element, and is the first element of a circular orbit and the first element after the handle for a ρ-shaped orbit. The orbit size $o$ of an orbit is the number of distinct elements in it. The handle size $h$ of an orbit is the number of elements in the orbit handle. The cycle size $c$ of an orbit is the number of elements in the orbit cycle.

**Lemma 2.4**  \[ o = h + c \]

**Lemma 2.5**  The distance from any point in an orbit to a point in a cycle of that orbit is always defined.

**Lemma 2.6**  If $x$ and $y$ are distinct points in a cycle of size $c$,

\[ c = \text{distance}(x, y, f) + \text{distance}(y, x, f) \]

**Lemma 2.7**  If $x$ and $y$ are points in a cycle of size $c$, the distance from $x$ to $y$ satisfies

\[ 0 \leq \text{distance}(x, y, f) < c \]
2.3 Collision Point

If we observe the behavior of a transformation, without access to its definition, we cannot determine whether a particular orbit is infinite: It might terminate or cycle back at any point. If we know that an orbit is finite, we can use an algorithm to determine the shape of the orbit. Therefore there is an implicit precondition of orbit finiteness for all the algorithms in this chapter.

There is, of course, a naive algorithm that stores every element visited and checks at every step whether the new element has been previously encountered. Even if we could use hashing to speed up the search, such an algorithm still would require linear storage and would not be practical in many applications. However, there is an algorithm that requires only a constant amount of storage.

The following analogy helps to understand the algorithm. If a fast car and a slow one start along a path, the fast one will catch up with the slow one if and only if there is a cycle. If there is no cycle, the fast one will reach the end of the path before the slow one. If there is a cycle, by the time the slow one enters the cycle, the fast one will already be there and will catch up eventually. Carrying our intuition from the continuous domain to the discrete domain requires care to avoid the fast one skipping past the slow one.\(^1\)

The discrete version of the algorithm is based on looking for a point where fast meets slow. The *collision point* of a transformation \(f\) and a starting point \(x\) is the unique \(y\) such that

\[
y = f^n(x) = f^{2n+1}(x)
\]

and \(n \geq 0\) is the smallest integer satisfying this condition. This definition leads to an algorithm for determining the orbit structure that needs one comparison of fast and slow per iteration. To handle partial transformations, we pass a definition-space predicate to the algorithm:

```cpp
template<typename F, typename P>
requires(Transformation(F) && UnaryPredicate(P) && Domain(F) == Domain(P))
Domain(F) collision_point(const Domain(F)& x, F f, P p)
{
    // Precondition: \(p(x) \iff f(x)\) is defined
    if (!p(x)) return x;
```

---

Domain(F) slow = x;  // slow = f^0(x)
Domain(F) fast = f(x);  // fast = f^1(x)

while (fast != slow) {
    slow = f(slow);  // slow = f^{n+1}(x) ∧ fast = f^{2n+1}(x)
    if (!p(fast)) return fast;
    fast = f(fast);  // slow = f^{n+1}(x) ∧ fast = f^{2n+2}(x)
    if (!p(fast)) return fast;
}
return fast;  // slow = f^n(x) ∧ fast = f^{2n+1}(x)

// Postcondition: return value is terminal point or collision point

We establish the correctness of collision_point in three stages: (1) verifying that it never applies \( f \) to an argument outside the definition space; (2) verifying that if it terminates, the postcondition is satisfied; and (3) verifying that it always terminates.

While \( f \) is a partial function, its use by the procedure is well defined, since the movement of \( \text{fast} \) is guarded by a call of \( p \). The movement of \( \text{slow} \) is unguarded, because by the regularity of \( f \), slow traverses the same orbit as fast, so \( f \) is always defined when applied to slow.

The annotations show that if, after \( n \geq 0 \) iterations, fast becomes equal to slow, then fast = \( f^{2n+1}(x) \) and slow = \( f^n(x) \). Moreover, \( n \) is the smallest such integer, since we checked the condition for every \( i < n \).

If there is no cycle, \( p \) will eventually return false because of finiteness. If there is a cycle, slow will eventually reach the connection point (the first element in the cycle). Consider the distance \( d \) from fast to slow at the top of the loop when slow first enters the cycle: \( 0 \leq d < c \). If \( d = 0 \), the procedure terminates. Otherwise the distance from fast to slow decreases by 1 on each iteration. Therefore the procedure always terminates; when it terminates, slow has moved a total of \( h + d \) steps.

The following procedure determines whether an orbit is terminating:

```cpp
template<typename F, typename P>
bool terminating(const Domain(F)& x, F f, P p)
```
2.3 Collision Point

```cpp
{  // Precondition: p(x) ⇔ f(x) is defined
    return !p(collision_point(x, f, p));
}
```

Sometimes we know either that the transformation is total or that the orbit is nonterminating for a particular starting element. For these situations it is useful to have a specialized version of `collision_point`:

```cpp
template<typename F>
    requires(Transformation(F))
Domain(F)
collision_point_nonterminating_orbit(const Domain(F)& x, F f)
{
    Domain(F) slow = x;   // slow = f^0(x)
    Domain(F) fast = f(x); // fast = f^1(x)
    // n ← 0 (completed iterations)
    while (fast != slow) {
        slow = f(slow);  // slow = f^n(x) ∧ fast = f^{2n+1}(x)
        fast = f(fast);  // fast = f^{2n+1}(x)
    }
    return fast;        // slow = f^n(x) ∧ fast = f^{2n+1}(x)
    // Postcondition: return value is collision point
}
```

In order to determine the cycle structure—handle size, connection point, and cycle size—we need to analyze the position of the collision point.

When the procedure returns the collision point

\[ f^n(x) = f^{2n+1}(x) \]

\( n \) is the number of steps taken by `slow`, and \( 2n + 1 \) is the number of steps taken by `fast`.

\[ n = h + d \]
where \( h \) is the handle size and \( 0 \leq d < c \) is the number of steps taken by slow inside the cycle. The number of steps taken by fast is
\[
2n + 1 = h + d + qc
\]
where \( q \geq 0 \) is the number of full cycles completed by fast when slow enters the cycle. Since \( n = h + d \),
\[
2(h + d) + 1 = h + d + qc
\]
Simplifying gives
\[
qc = h + d + 1
\]
Let us represent \( h \) modulo \( c \):
\[
h = mc + r
\]
with \( 0 \leq r < c \). Substitution gives
\[
qc = mc + r + d + 1
\]
or
\[
d = (q - m)c - r - 1
\]
\( 0 \leq d < c \) implies
\[
q - m = 1
\]
so
\[
d = c - r - 1
\]
and \( r + 1 \) steps are needed to complete the cycle.
Therefore the distance from the collision point to the connection point is
\[
e = r + 1
\]
In the case of a circular orbit \( h = 0, r = 0 \), and the distance from the collision point to the beginning of the orbit is
\[
e = 1
\]
### 2.3 Collision Point

Circularity, therefore, can be checked with the following procedures:

```cpp
template<typename F>
    requires(Transformation(F))
bool circular_nonterminating_orbit(const Domain(F)& x, F f)
{
    return x == f(collision_point_nonterminating_orbit(x, f));
}

template<typename F, typename P>
    requires(Transformation(F) && UnaryPredicate(P) &&
     Domain(F) == Domain(P))
bool circular(const Domain(F)& x, F f, P p)
{
    // Precondition: p(x) ⇔ f(x) is defined
    Domain(F) y = collision_point(x, f, p);
    return p(y) && x == f(y);
}
```

We still don’t know the handle size $h$ and the cycle size $c$. Determining the latter is simple once the collision point is known: Traverse the cycle and count the steps.

To see how to determine $h$, let us look at the position of the collision point:

$$f^{h+d}(x) = f^{h+c-r-1}(x) = f^{mc+r+c-r-1}(x) = f^{(m+1)c-1}(x)$$

Taking $h + 1$ steps from the collision point gets us to the point $f^{(m+1)c+h}(x)$, which equals $f^h(x)$, since $(m + 1)c$ corresponds to going around the cycle $m + 1$ times. If we simultaneously take $h$ steps from $x$ and $h + 1$ steps from the collision point, we meet at the connection point. In other words, the orbits of $x$ and $1$ step past the collision point converge in exactly $h$ steps, which leads to the following sequence of algorithms:

```cpp
template<typename F>
    requires(Transformation(F))
Domain(F) convergent_point(Domain(F) x0, Domain(F) x1, F f)
{
    while (x0 != x1) {
```
\[ x_0 = f(x_0); \]
\[ x_1 = f(x_1); \]

\}
return x0;
}\}

template<typename F>
  requires(Transformation(F))
Domain(F)
connection_point_nonterminating_orbit(const Domain(F)& x, F f)
{
  return convergent_point(
    x,
    f(collision_point_nonterminating_orbit(x, f)),
    f);
}

template<typename F, typename P>
  requires(Transformation(F) && UnaryPredicate(P) &&
            Domain(F) == Domain(P))
Domain(F) connection_point(const Domain(F)& x, F f, P p)
{
  // Precondition: p(x) \Leftrightarrow f(x) is defined
  Domain(F) y = collision_point(x, f, p);
  if (!p(y)) return y;
  return convergent_point(x, f(y), f);
}

**Lemma 2.8** If the orbits of two elements intersect, they have the same cyclic elements.

**Exercise 2.2** Design an algorithm that determines, given a transformation and its definition-space predicate, whether the orbits of two elements intersect.

**Exercise 2.3** For convergent_point to work, it must be called with elements whose distances to the convergent point are equal. Implement an algorithm convergent_point_guarded for use when that is not known to be the case, but there is an element in common to the orbits of both.
2.4 Measuring Orbit Sizes

The natural type to use for the sizes \(o\), \(h\), and \(c\) of an orbit on type \(T\) would be an integer count type large enough to count all the distinct values of type \(T\). If a type \(T\) occupies \(k\) bits, there can be as many as \(2^k\) values, so a count type occupying \(k\) bits could not represent all the counts from 0 to \(2^k\). There is a way to represent these sizes by using distance type.

An orbit could potentially contain all values of a type, in which case \(o\) might not fit in the distance type. Depending on the shape of such an orbit, \(h\) and \(c\) would not fit either. However, for a \(\rho\)-shaped orbit, both \(h\) and \(c\) fit. In all cases each of these fits: \(o - 1\) (the maximum distance in the orbit), \(h - 1\) (the maximum distance in the handle), and \(c - 1\) (the maximum distance in the cycle). That allows us to implement procedures returning a triple representing the complete structure of an orbit, where the members of the triple are as follows:

<table>
<thead>
<tr>
<th>Case</th>
<th>m0</th>
<th>m1</th>
<th>m2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Terminating</td>
<td>(h - 1)</td>
<td>0</td>
<td>terminal element</td>
</tr>
<tr>
<td>Circular</td>
<td>0</td>
<td>(c - 1)</td>
<td>(x)</td>
</tr>
<tr>
<td>(\rho)-shaped</td>
<td>(h)</td>
<td>(c - 1)</td>
<td>connection point</td>
</tr>
</tbody>
</table>

\[
\text{template<typename } F\rangle
\]
\[
\text{requires(Transformation}(F))}
\[
\text{triple\langle DistanceType}(F), \text{DistanceType}(F), \text{Domain}(F)\rangle
\]
\[
\text{orbit\_structure\_nonterminating\_orbit(const Domain}(F) & x, F f)\}
\{
    typedef DistanceType(F) N;
    Domain(F) y = connection\_point\_nonterminating\_orbit(x, f);
    return triple\langle N, N, Domain(F)\rangle(distance(x, y, f),
                                                  distance(f(y), y, f),
                                                  y);
\}

\[
\text{template<typename } F, \text{typename } P\rangle
\]
\[
\text{requires(Transformation}(F) \&\&
    \text{UnaryPredicate}(P) \&\& \text{Domain}(F) == \text{Domain}(P))}
\[
\text{triple\langle DistanceType}(F), \text{DistanceType}(F), \text{Domain}(F)\rangle
\]
\[
\text{orbit\_structure(const Domain}(F) & x, F f, P p)\}
\{
    // Precondition: p(x) \Leftrightarrow f(x) is defined
typedef DistanceType(F) N;
Domain(F) y = connection_point(x, f, p);
N m = distance(x, y, f);
N n(0);
if (p(y)) n = distance(f(y), y, f);
// Terminating: m = h − 1 ∧ n = 0
// Otherwise: m = h ∧ n = c − 1
return triple<N, N, Domain(F)>(m, n, y);
}

Exercise 2.4  Derive formulas for the count of different operations (f, p, equality) for the algorithms in this chapter.

Exercise 2.5  Use orbit_structure_nonterminating_orbit to determine the average handle size and cycle size of the pseudorandom number generators on your platform for various seeds.

2.5 Actions

Algorithms often use a transformation f in a statement like

x = f(x);

Changing the state of an object by applying a transformation to it defines an action on the object. There is a duality between transformations and the corresponding actions: An action is definable in terms of a transformation, and vice versa:

void a(T& x) { x = f(x); }  // action from transformation

and

T f(T x) { a(x); return x; }  // transformation from action

Despite this duality, independent implementations are sometimes more efficient, in which case both action and transformation need to be provided. For example, if a transformation is defined on a large object and modifies only part of its overall state, the action could be considerably faster.

Exercise 2.6  Rewrite all the algorithms in this chapter in terms of actions.
2.6 Conclusions

**Project 2.1** Another way to detect a cycle is to repeatedly test a single advancing element for equality with a stored element while replacing the stored element at ever-increasing intervals. This and other ideas are described in Sedgewick, et al. [1979], Brent [1980], and Levy [1982]. Implement other algorithms for orbit analysis, compare their performance for different applications, and develop a set of recommendations for selecting the appropriate algorithm.

2.6 Conclusions

Abstraction allowed us to define abstract procedures that can be used in different domains. Regularity of types and functions is essential to make the algorithms work: fast and slow follow the same orbit because of regularity. Developing nomenclature is essential (e.g., orbit kinds and sizes). Affiliated types, such as distance type, need to be precisely defined.
Index

→ (function), 231
− (additive inverse), in additive group, 67
∧ (and), 231
− (difference)
in additive group, 67
in cancellable monoid, 72
of integers, 18
of iterator and integer, 111
of iterators, 93
× (direct product), 231
∈ (element), 231
= (equality), 7
for array_k, 212
for pair, 210
≜ (equals by definition), 12, 231
⇔ (equivalent), 231
∃ (exists), 231
∀ (for all), 231
> (greater), 62
≥ (greater or equal), 62
⇒ (implies), 231
[ ] (index)
for array_k, 211
for bounded_range, 214
≠ (inequality), 7, 62
∩ (intersection), 231
< (less), 62
for array_k, 212
natural total ordering, 61
for pair, 210
≤ (less or equal), 62
↪ (maps to), 231
¬ (not), 231
∨ (or), 231
a^n (power of associative operation), 32
f^n (power of transformation), 17
≺ (precedes), 95
≤ (precedes or equal), 95
· (product)
of integers, 18
in multiplicative semigroup, 66
in semimodule, 69
/ (quotient), of integers, 18
[f, l] (range, closed bounded), 94
[f, n] (range, closed weak or counted), 94
[f, l) (range, half-open bounded), 94
[f, n) (range, half-open weak or counted), 94
⊂ (subset), 231
+ (sum)
in additive semigroup, 66
of integers, 18
of iterator and integer, 92
∪ (union), 231

A
abs algorithm, 16, 71
absolute value, properties, 71
abstract entity, 1
abstract genus, 2
abstract procedure, 13
overloading, 43
abstract species, 2
accumulation procedure, 46
accumulation variable
  elimination, 39
  introduction, 35
action, 28
acyclic descendants of bifurcate coordinate, 116
additive inverse (−), in additive group, 67
AdditiveGroup concept, 67
AdditiveMonoid concept, 67
AdditiveSemigroup concept, 66
address, 4
abstracted by iterator, 89
add_to_counter algorithm, 199
advance_tail machine, 135
algorithm. See machine
abs, 16, 71
add_to_counter, 199
all, 97
bifurcate_compare, 131
bifurcate_compare_nonempty, 130
bifurcate_equivalent, 129
bifurcate_equivalent_nonempty, 128
bifurcate_isomorphic, 126
bifurcate_isomorphic_nonempty, 125
circular, 25
circular_nonterminating_orbit, 25
collision_point, 22
collision_point_nonterminating_orbit, 23
combine_copy, 160
combine_copy_backward, 162
combine_linked_nonempty, 138
combine_ranges, 196
compare_strict_or_reflexive, 57–58
complement, 50
complement_of_converse, 50
connection_point, 26
connection_point_nonterminating_orbit, 26
convergent_point, 26
converse, 50
copy, 152
copy_backward, 155
copy_bounded, 153
copy_if, 158
copy_n, 154
copy_select, 158
count_if, 97, 98
cycle_from, 173
cycle_to, 173
distance, 19
euclidean_norm, 16
exchange_values, 164
fast_subtractive_gcd, 78
fibonacci, 46
find, 96
find_adjacent_mismatch, 103
find_adjacent_mismatch_forward, 106, 135
find_backward_if, 112
find_if, 97
find_if_not_unguarded, 102
find_if_unguarded, 101
find_last, 136
find_mismatch, 102
find_n, 101
find_not, 97
for_each, 96
for_each_n, 101
gcd, 80
height, 122
height_recursive, 118
increment, 91
is_left_successor, 119
is_right_successor, 120
k_rotate_from_permutation_indexed, 180
k_rotate_from_permutation_random_access, 180
largest_doubling, 75
lexicographical_compare, 129
lexicographical_equal, 127
lexicographical_equivalent, 127
lexicographical_less, 130
lower_bound_n, 109
lower_bound_predicate, 108
median_5, 61
memory-adaptive, 177
merge_copy, 163
merge_copy_backward, 163
merge_linked_nonempty, 141
merge_n_adaptive, 206
merge_n_with_buffer, 202
none, 97
not_all, 97
orbit_structure, 28
orbit_structure_nonterminating_orbit, 27
partitioned_at_point, 191
partition_bidirectional, 194
partition_copy, 160
partition_copy_n, 160
partition_linked, 140
partition_point, 107
partition_point_n, 107
partition_semistable, 192
partition_single_cycle, 194
partition_stable_iterative, 201
partition_stable_n, 197
partition_stable_n_adaptive, 197
partition_stable_n_nonempty, 197
algorithm. See machine (cont.)
partition_stable_singleton, 196
partition_stable_with_buffer, 195
partition_trivial, 198
phased_applicator, 147
potential_partition_point, 191
power, 42
power_accumulate, 41
power_accumulate_positive, 41
power_left_associated vs. power_0, 34
power_right_associated, 33
power Unary, 18
predicate_source, 140
quotient_remainder, 85
quotient_remainder_nonnegative, 82
quotient_remainder_nonnegative_iterative, 83
reachable, 121
reduce, 99
reduce_balanced, 200
reduce_nonempty, 99
reduce_nonzeroes, 100
relation_source, 141
remainder, 84
remainder_nonnegative, 74
remainder_nonnegative_iterative, 75
reverse_append, 139, 140
reverse_bidirectional, 175
reverse_copy, 156
reverse_copy_backward, 156
reverse_indexed, 186
reverse_n_adaptive, 178
reverse_n_bidirectional, 175
reverse_n_forward, 177
reverse_n_indexed, 175
reverse_n_with_buffer, 176
reverse_swap_ranges, 167
reverse_swap_ranges_bounded, 167
reverse_swap_ranges_n, 168
reverse_with_temporary_buffer, 187, 225
rotate, 187
rotate_bidirectional_nontrivial, 182
rotate_cycles, 181
rotate_forward.annotated, 183
rotate_forward_nontrivial, 184
rotate_forward_step, 184
rotate_indexed_nontrivial, 181
rotate_nontrivial, 188
rotate_partial_nontrivial, 185
rotate_random_access_nontrivial, 181
rotate_with_buffer_backward_nontrivial, 186
rotate_with_buffer_nontrivial, 185
select.0.2, 53, 63
select.0.3, 54
select.1.2, 54
select.1.3, 55
select.1.3_ab, 55
select.1.4, 56, 59
select.1.4_ab, 56, 59
select.1.4_ab_cd, 56, 58
select.2.3, 54
select.2.5, 60
select.2.5_ab, 60
select.2.5_ab_cd, 56, 58
slow_quotient, 73
slow_remainder, 72
some, 97
sort_linked_nonempty_n, 142
sort_n, 207
sort_n_adaptive, 207
sort_n_with_buffer, 203
split_copy, 158
split_linked, 137
subtractive_gcd, 78
subtractive_gcd_nonzero, 77
swap, 224
swap_basic, 223
swap_ranges, 165
swap_ranges_bounded, 166
swap_ranges_n, 166
terminating, 23
transpose_operation, 201
traverse, 123
traverse_nonempty, 118
traverse_phased_rotating, 148
traverse_rotating, 146
underlying_ref, 224
upper_bound_n, 109
upper_bound_predicate, 109
weight, 122
weight_recursive, 117
weight_rotating, 147
aliased property, 150
aliased write-read, 150
aliased write-write, 159
all algorithm, 97
ambiguous value type, 3
amortized complexity, 219
and \((\wedge)\), 231
annihilation property, 68
annotation variable, 183
ArchimedeanGroup concept, 83
ArchimedeanMonoid concept, 72
area of object, 227
Aristotle, 77
Arity type attribute, 11
array, varieties, 220–221
array\_k type, 210
Artin, Emil, 13
assignment, 7
for array\_k, 211
for pair, 210
associative operation, 31, 98
power of \((\alpha^n)\), 32
associative property, 31
exploited by power, 33
partially_associative, 98
of permutation composition, 170
asymmetric property, 50
attribute, 1
auxiliary computation during recursion, 176
Axiom of Archimedes, 72, 73

B
backward movement in range, 112
BackwardLinker concept, 134
backward_offset property, 161
basic singly linked list, 218
begin
for array\_k, 211
for bounded\_range, 214
for Linearizable, 213
behavioral equality, 3, 228
BidirectionalBifurcateCoordinate concept, 119–120
BidirectionalIterator concept, 111
BidirectionalLinker concept, 134
BifurcateCoordinate concept, 115
bifurcate\_compare algorithm, 131
bifurcate\_compare\_nonempty algorithm, 130
bifurcate\_equivalent algorithm, 129
bifurcate\_equivalent\_nonempty algorithm, 128
bifurcate\_isomorphic algorithm, 126
bifurcate\_isomorphic\_nonempty algorithm, 125
BinaryOperation concept, 31
binary\_scale\_down\_nonnegative, 41
binary\_scale\_up\_nonnegative, 41
bisection technique, 107
Bolzano, Bernard, 107
bounded integer type, 87
bounded range, 93
bounded\_range property, 93
bounded\_range type, 214
Brandt, Jon, 193

C
CancellableMonoid concept, 72
cancellation in monoid, 72
categories of ideas, 1
Cauchy, Augustin Louis, 107
circular algorithm, 25
circular array, 220
circular doubly linked list, 218
circular singly linked list, 218
circular\_nonterminating\_orbit algorithm, 25
closed bounded range \(([f, l])\), 94
closed interval, 231
closed weak or counted range \(([f, n])\), 94
clusters of derived procedures, 62
codomain, 10
Codomain type function, 11
Collins, George, 13
collision point of orbit, 21
collision\_point algorithm, 22
collision\_point\_nonterminating\_orbit algorithm, 23
combine\_copy algorithm, 160
combine\_copy\_backward algorithm, 162
combine\_linked\_nonempty algorithm, 138
combine\_ranges algorithm, 196
common-subexpression elimination, 35
commutative property, 66
CommutativeRing concept, 69
CommutativeSemiring concept, 68
compare\_strict\_or\_reflexive algorithm, 57–58
complement algorithm, 50
complement of converse of relation, 50
complement of relation, 50
Index

complement_of_converse algorithm, 50
complement_of_converse property, 104
complexity
  amortized, 219
  of empty, 213
  of indexing of a sequence, 213
  of regular operations, 227
  of source, 90
  of successor, 92
composite object, 215
composition
  of permutations, 170
  of transformations, 17, 32
computational basis, 6
concept, 11
  AdditiveGroup, 67
  AdditiveMonoid, 67
  AdditiveSemigroup, 66
  ArchimedeanGroup, 83
  ArchimedeanMonoid, 72
  BackwardLinker, 134
  BidirectionalBifurcateCoordinate, 119–120
  BidirectionalIterator, 111
  BidirectionalLinker, 134
  BifurcateCoordinate, 115
  BinaryOperation, 31
  CancellableMonoid, 72
  CommutativeRing, 69
  CommutativeSemiring, 68
  consistent, 87
  DiscreteArchimedeanRing, 86
  DiscreteArchimedeanSemiring, 85
  EmptyLinkedBifurcateCoordinate, 144
  EuclideanMonoid, 77
  EuclideanSemimodule, 80
  EuclideanSemiring, 79
  examples from C++ and STL, 11
  ForwardIterator, 106
  ForwardLinker, 133
  FunctionalProcedure, 11
  HalvableMonoid, 74
  HomogeneousFunction, 12
  HomogeneousPredicate, 16
  IndexedIterator, 110
  Integer, 18, 40
  Iterator, 91
  Linearizable, 213
  LinkedBifurcateCoordinate, 144
modeled by type, 11
Module, 70
MultiplicativeGroup, 68
MultiplicativeMonoid, 67
MultiplicativeSemigroup, 66
NonnegativeDiscreteArchimedeanSemiring, 86
Operation, 16
OrderedAdditiveGroup, 70
OrderedAdditiveMonoid, 70
OrderedAdditiveSemigroup, 70
Predicate, 15
RandomAccessIterator, 113
refinement, 11
Regular, 11
Relation, 49
relational concept, 69
Ring, 69
Semimodule, 69
Semiring, 68
Sequence, 216
TotallyOrdered, 62
Transformation, 17
type concept, 11
UnaryFunction, 12
UnaryPredicate, 16
univalent, 86
useful, 87
weakening, 11
concept dispatch, 106, 187
concept schema
  composite object, 216
  coordinate structure, 124
concept tag type, 187
cornerstone entity, 1
cornerstone genus, 2
cornerstone species, 2
connectedness of composite object, 215
connection point of orbit, 20
connection_point algorithm, 26
connection_point_nonterminating_orbit algorithm, 26
connectors, 229
consistency of concept’s axioms, 87
constant-size sequence, 216
constructor, 7
container, 213
convergent_point algorithm, 26
converse algorithm, 50
converse of relation, 50
coordinate structure
  bifurcate coordinate, 115
  of composite object, 215
  concept schema, 124
  iterator, 89
copy algorithm, 152
copy constructor, 8
  for array_k, 211
  for pair, 210
copy of object, 5
copying rearrangement, 172
copy_backward algorithm, 155
copy_backward_step machine, 154
copy_bounded algorithm, 153
copy_if algorithm, 158
copy_in algorithm, 154
copy_select algorithm, 158
copy_step machine, 152
counted_range property, 93
counter_machine type, 200
count_down machine, 153
count_if algorithm, 97, 98
cycle detection intuition, 21
cycle in a permutation, 171
cycle of orbit, 20
cycle size, 20
cycle_from algorithm, 173
cycle_to algorithm, 173
cyclic element under transformation, 18
  cyclic permutation, 171

D
DAG (directed acyclic graph), 116
datum, 2
de Bruijn, N. G., 74
default constructor, 8
  for array_k, 211
  for pair, 209
default ordering, 62
default total ordering, 62
  importance of, 228
definition space, 9
definition-space predicate, 17
dependence of axiom, 86
deref, 150
derived relation, 50
descendant of bifurcate coordinate, 116
destructor, 7
  for pair, 210
difference (−)
  in additive group, 67
  in cancellable monoid, 72
  of integers, 18
  of iterator and integer, 111
  of iterators, 93
DifferenceType type function, 113
direct product (×), 231
directed acyclic graph, 116
DiscreteArchimedeanRing concept, 86
DiscreteArchimedeanSemiring concept, 85
discreteness property, 85
disjoint property, 134
disjointness of composite object, 216
distance algorithm, 19
distance in orbit, 19
DistanceType type function, 17, 91
distributive property, holds for semiring, 68
divisibility on an Archimedean monoid, 76
division, 68
domain, 10
Domain type function, 12
double-ended array, 220
doubly linked list, 218–219
Dudziński, Krzysztof, 206
dummy node doubly linked list, 218
Dydek, Andrzej, 206
dynamic-size sequence, 216

E
efficient computational basis, 6
element (∈), 231
eliminating common subexpression, 35
empty
  for array_k, 212
  for bounded_range, 214
  for Linearizable, 213
empty coordinate, 144
empty range, 95
EmptyLinkedBifurcateCoordinate concept, 144
end
for array_k, 211
for bounded_range, 214
for Linearizable, 213
entity, 1
equality
=, 7
≠, 62
for array_k, 212
behavioral, 3, 228
equal for Regular, 127
for objects, 5
for pair, 210
for regular type, 7
representational, 3, 228
structural, 228
for uniquely represented type, 3
for value type, 3
equals by definition (≜), 12, 231
equational reasoning;: 4
equivalence class, 51
equivalence property, 51
equivalent (⇔), 231
equivalent coordinate collections, 126
erasure in a sequence, 217
Euclidean function, 79
EuclideanMonoid concept, 77
EuclideanSemimodule concept, 80
EuclideanSemiring concept, 79
euclidean_norm algorithm, 16
even, 41
exchange_values algorithm, 164
exists (∃), 231
expressive computational basis, 6

F
fast_subtractive_gcd algorithm, 78
fibonacci algorithm, 46
Fibonacci sequence, 45
find algorithm, 96
find_adjacent_mismatch algorithm, 103
find_adjacent_mismatch_forward algorithm, 106, 135
find_backward_if algorithm, 112
find_if algorithm, 97
find_if_not, 97
find_if_not_unguarded algorithm, 102
find_if_unguarded algorithm, 101
find_last algorithm, 136
find_mismatch algorithm, 102
find_n algorithm, 101
find_not algorithm, 97
finite order, under associative operation, 32
finite set, 171
first-last singly linked list, 218
fixed point of transformation, 170
fixed-size sequence, 216
Floyd, Robert W., 21
for all (∀), 231
ForwardIterator concept, 106
ForwardLinker concept, 133
forward_offset property, 162
for_each algorithm, 96
for_each_n algorithm, 101
Frobenius, Georg Ferdinand, 32
from-permutation, 172
function, 2
→, 231
on abstract entities, 2
on values, 3
function object, 9, 96, 236
functional procedure, 9
FunctionalProcedure concept, 11

G
garbage collection, 230
Gaussian integers, 40
Stein’s algorithm, 81
gcd, 76
Stein, 81
subtractive, 76
gcd algorithm, 80
genus, 2
global state, 6
goto statement, 148
greater (>), 62
greater or equal (≥), 62
greatest common divisor (gcd), 76
group, 67
of permutations, 170

H
half_nonnegative, 41
half-open bounded range ([f, l]), 94
half-open interval, 231
half-open weak or counted range ([f, n]), 94
HalvableMonoid concept, 74
handle of orbit, 20
handle size, 20
header of composite object, 217
height algorithm, 122
height of bifurcate coordinate (DAG), 116
height_recursive algorithm, 118
Ho, Wilson, 182
Hoare, C. A. R., 195
homogeneous functional procedure, 10
HomogeneousFunction concept, 12
HomogeneousPredicate concept, 16

I
ideas, categories of, 1
identity
of concrete entity, 1
of object, 5
identity element, 65
identity token, 5
identity transformation, 170
identity_element property, 65
implies (⇒), 231
inconsistency of concept, 87
increasing range, 103
increasing_counted_range property, 105
increasing_range property, 105
increment algorithm, 91
independence of proposition, 86
index ([ ])
for array_k, 211
for bounded_range, 214
index permutation, 172
index of segmented array, 221
indexed iterator
equivalent to random-access iterator, 113
IndexedIterator concept, 110
inequality (≠), 7
standard definition, 62
inorder, 118
input object, 6
input/output object, 6
InputType type function, 11
insertion in a sequence, 217
Integer concept, 18, 40
interpretation, 2
intersection (∩), 231
interval, 231
into transformation, 169
invariant, 148
loop, 37
recursion, 36
inverse of permutation, 170, 171
inverse_operation property, 66
isomorphic coordinate sets, 124
isomorphic types, 86
is_left_successor algorithm, 119
is_right_successor algorithm, 120
iterator adapter
for bidirectional bifurcate coordinates, project, 124
random access from indexed, 114
reverse from bidirectional, 112
underlying type, 224
Iterator concept, 91
iterator invalidation in array, 221
IteratorConcept type function, 187
IteratorType type function, 133, 134, 213

K
Kislitsyn, Sergei, 55
k_rotate_from_permutation_indexed algorithm, 180
k_rotate_from_permutation_random_access algorithm, 180

L
Lagrange, J.-L., 107
Lakshman, T. K., 159
largest_doubling algorithm, 75
less (<), 62
for array_k, 212
for bounded_range, 215
less for TotallyOrdered, 130
natural total ordering, 61
for pair, 210
less or equal (≤), 62
lexicographical_compare algorithm, 129
lexicographical_equal algorithm, 127
lexicographical Equivalent algorithm, 127
lexicographical_less algorithm, 130
limit in a range, 95
linear ordering, 52
Linearizable concept, 213
link rearrangement, 134
on lists, 219
linked iterator, 133
linked structures, forward vs. bidirectional, 219
LinkedBifurcateCoordinate concept, 144
linker object, 133
linker_to_head machine, 139
linker_to_tail machine, 135
links, reversing, 145
list
doubly linked, 218
singly linked, 218
Lo, Raymond, 182
load, 4
local part of composite object, 217
local state, 6
locality of reference, 143
loop invariant, 37
lower bound, 107
lower_bound algorithm, 109
lower_bound predicate algorithm, 108

M
machine, 120
  advance_tail, 135
  copy_backward_step, 154
  copy_step, 152
  count_down, 153
  linker_to_head, 139
  linker_to_tail, 135
merge_n_step_0, 205
merge_n_step_1, 205
reverse_copy_backward_step, 156
reverse_copy_step, 155
reverse_swap_step, 166
swap_step, 165
traverse_step, 121
tree_rotate, 145
maps to (→), 231
marking, 118
Mauchly, John W., 107
median_5 algorithm, 61
memory, 4
memory-adaptive algorithm, 177
merge, stability, 203
mergeable property, 203
merge_copy algorithm, 163
merge_copy_backward algorithm, 163
merge_linked_nonempty algorithm, 141
merge_n_adaptive algorithm, 206
merge_n_step_0 machine, 205
merge_n_step_1 machine, 205
merge_n_with_buffer algorithm, 202
mod (remainder), 18
model, partial, 70
models, 11
Module concept, 70
monoid, 67
multipass traversal, 106
MultiplicativeGroup concept, 68
MultiplicativeMonoid concept, 67
MultiplicativeSemigroup concept, 66
multiset, 227
Musser, David, 13
mutable range, 151
mutable_bounded_range property, 151
mutable_counted_range property, 151
mutable_weak_range property, 151
mutative rearrangement, 172

N
natural total ordering, < reserved for, 61
negative, 41
nil, 134
Noether, Emmy, 13
noncircularity of composite object, 216
none algorithm, 97
NonnegativeDiscreteArchimedeanSemiring concept, 86
nontotal procedure, 17
not (¬), 231
not_all algorithm, 97
not_overlapped property, 157
not_overlapped_backward property, 155
not_overlapped_forward property, 153
not_write_overlapped property, 159
null link, 218

O
object, 4
  area, 227
  equality, 5
  starting address, 216
  state, 4
object type, 4
odd, 41
one, 41
one-to-one transformation, 169
onto transformation, 169
open interval, 231
Operation concept, 16
or (\(\lor\)), 231
orbit, 18–20
orbit structure algorithm, 28
orbit structure nonterminating orbit algorithm, 27
OrderedAdditiveGroup concept, 70
OrderedAdditiveMonoid concept, 70
OrderedAdditiveSemigroup concept, 70
ordering, linear, 52
ordering-based rearrangement, 172
output object, 6
overloading, 43, 133, 144
own state, 6
ownership, of parts by composite object, 216

P
pair type, 11, 209
parameter passing, 9
part of composite object, 215–219
partial model, 70
partial procedure, 17
partial (usage convention), 232
partially formed object state, 7
partially associative property, 98
partition algorithm, origin of, 195
partition point, 105
  lower and upper bounds, 107
partition rearrangement, semistable, 192
partitioned property, 105
partitioned range, 105
partitioned at point algorithm, 191
partition bidirectional algorithm, 194
partition copy algorithm, 160
partition copy n algorithm, 160
partition linked algorithm, 140
partition point algorithm, 107
partition point n algorithm, 107
partition semistable algorithm, 192
partition single cycle algorithm, 194
partition stable iterative algorithm, 201
partition stable n algorithm, 197
partition stable n adaptive algorithm, 197
partition stable n nonempty algorithm, 197
partition stable singleton algorithm, 196
partition stable with buffer algorithm, 195
partition trivial algorithm, 198
permanently placed part of composite object, 217
permutation, 170
  composition, 170
cycle, 171
cyclic, 171
from, 172
index, 172
inverse, 170, 171
product of its cycles, 171
reverse, 174
rotation, 178
to, 172
transposition, 171
permutation group, 170
phased applicator algorithm, 147
pivot, 205
position-based rearrangement, 172
positive, 41
postorder, 118
potential partition point algorithm, 191
power
  of associative operation (\(a^n\)), 32
  powers of same element commute, 32
  of transformation (\(f^n\)), 17
power algorithm, 42
  operation count, 34
power accumulate algorithm, 41
power accumulate positive algorithm, 41
power right associated algorithm, 33
power unary algorithm, 18
precedence preserving link rearrangement, 135
precedes (\(<\)), 95
precedes or equal (\(\leq\)), 95
precondition, 13
predecessor
  of integer, 41
  of iterator, 111
Predicate concept, 15
predicate-based rearrangement, 172
predicate source algorithm, 140
prefix of extent, 220
preorder, 118
prime property, 14
procedure, 6
  abstract, 13
  functional, 9
  nontotal, 17
  partial, 17
  total, 17
product (·)
  of integers, 18
  in multiplicative semigroup, 66
  in semimodule, 69
program transformation
  accumulation-variable elimination, 39
  accumulation-variable introduction, 35
  common-subexpression elimination, 35
  enabled by regular types, 35
  forward to backward iterators, 112
  relaxing precondition, 38
  strengthening precondition, 38
  strict tail-recursive, 37
  tail-recursive form, 35
project
  abstracting platform-specific copy
    algorithms, 164
  algorithms for bidirectional bifurcate
    algorithms, 123
  axioms for random-access iterator, 113
  benchmark and composite algorithm for
    rotate, 189
  concepts for bounded binary
    integers, 87
  coordinate structure concept, 131
  cross-type operations, 14
  cycle-detection algorithms, 29
  dynamic-sequences benchmark, 222
  dynamic-sequences implementation, 222
  dynamic-sequences interfaces, 222
  floating-point nonassociativity, 42
  isomorphism, equivalence, and ordering
    using tree_rotate, 148
  iterator adapter for bidirectional bifurcate
    coordinates, 124
  linear recurrence sequences, 47
  minimum-comparison stable sorting and
    merging, 61
  nonhalvable Archimedean monoids, 75
  order-selection stability, 61
  reallocation strategy for single-extent
    arrays, 221
  searching for a subsequence within a
    sequence, 114
  setting for Stein gcd, 81
  sorting library, 208
  underlying type used in major library, 225
  projection regularity, 216
  proper underlying type, 223
  properly partial object state, 5
  properly partial value type, 2
property
  aliased, 150
  annihilation, 68
  associative, 31
  asymmetric, 50
  backward_offset, 161
  bounded_range, 93
  commutative, 66
  complement_of_converse, 104
  counted_range, 93
  discreteness, 85
  disjoint, 134
  distributive, 68
  equivalence, 51
  forward_offset, 162
  identity_element, 65
  identity_element, 65
  increasing_counted_range, 105
  increasing_range, 105
  inverse_operation, 66
  mergeable, 203
  mutable_bounded_range, 151
  mutable_counted_range, 151
  mutable_weak_range, 151
  notation, 14
  not_overlapped, 157
  not_overlapped_backward, 155
  not_overlapped_forward, 153
  not_write_overlapped, 159
  partially_associative, 98
  partitioned, 105
  prime, 14
  readable_bounded_range, 95
  readable_counted_range, 96
  readable_tree, 123
  readable_weak_range, 96
  reflexive, 50
  regular_unary_function, 14
  relation_preserving, 103
property (cont.)
  strict, 50
  strictly_increasing_counted_range, 105
  strictly_increasing_range, 104
  symmetric, 50
  total_ordering, 51
  transitive, 49
  tree, 117
  trichotomy, 51
  weak_trichotomy, 51
  weak_ordering, 52
  writable_range, 92
  writable_bounded_range, 150
  writable_counted_range, 150
  writable_weak_range, 150
  write_aliased, 159
proposition, independence of, 86
pseudopredicate, 136
pseudorelation, 137
pseudotransformation, 91

Q
quotient (/), of integers, 18
quotient
  in Euclidean semimodule, 80
  in Euclidean semiring, 79
QuotientType type function, 72
quotient_remainder algorithm, 85
quotient_remainder_nonnegative algorithm, 82
quotient_remainder_nonnegative_iterative algorithm, 83

R
random-access iterator, equivalent to indexed iterator, 113
RandomAccessIterator concept, 113
range
  backward movement, 112
  closed bounded ([f, l]), 94
  closed weak or counted ([f, n]), 94
  empty, 95
  half-open bounded ([f, l]), 94
  half-open weak or counted ([f, n]), 94
  increasing, 103
  limit, 95
  lower bound, 107
mutable, 151
partition point, 105
partitioned, 105
readable, 95
size, 94
strictly increasing, 103
upper bound, 107
writable, 150
reachability
  of bifurcate coordinate, 116
  in orbit, 18
reachable algorithm, 121
readable range, 95
readable_bounded_range property, 95
readable_counted_range property, 96
readable_tree property, 123
readable_weak_range property, 96
rearrangement, 172
  bin-based, 172
  copying, 172
  link, 134
  mutative, 172
  ordering-based, 172
  position-based, 172
  reverse, 174
  rotation, 179
recursion invariant, 36
reduce algorithm, 99
reduce_balanced algorithm, 200
reduce_nonempty algorithm, 99
reduce_nonzeroes algorithm, 100
reduction, 98
reference counting, 230
refinement of concept, 11
reflexive property, 50
Regular concept, 11
  and program transformation, 35
regular function on value type, 3
regular type, 6–8
regularity, 216, 217
regular_unary_function property, 14
Relation concept, 49
relational concept, 69
relationship, 229
relation_preserving property, 103
relation_source algorithm, 141
relaxing precondition, 38
remainder
algorithm, 84
  in Euclidean semimodule, 80
  in Euclidean semiring, 79
remainder (mod), of integers, 18
remainder_nonnegative algorithm, 74
remainder_nonnegative_iterative algorithm, 75
remote part of composite object, 217
representation, 2
representation equality, 3, 228
requires clause, 13
  syntax, 240
resources, 4
result space, 10
returning useful information, 87, 96, 97,
  101–103, 106, 112, 152, 153, 159,
  163, 174, 179, 182, 211
reverse rearrangement, 174
reverse_append algorithm, 139, 140
reverse_bidirectional algorithm, 175
reverse_copy algorithm, 156
reverse_copy_backward algorithm, 156
reverse_copy_backward_step machine, 156
reverse_copy_step machine, 155
reverse_indexed algorithm, 186
reverse_n_adaptive algorithm, 178
reverse_n_bidirectional algorithm, 175
reverse_n_forward algorithm, 177
reverse_n_indexed algorithm, 175
reverse_n_with_buffer algorithm, 176
reverse_swap_ranges algorithm, 167
reverse_swap_ranges_bounded
  algorithm, 167
reverse_swap_ranges_n algorithm, 168
reverse_swap_step machine, 166
reverse_with_temporary_buffer algorithm,
  187, 225
reversing links, 145
Rhind Mathematical Papyrus
  division, 73
  power, 33
Ring concept, 69
rotate algorithm, 187
rotate_bidirectional_nontrivial
  algorithm, 182
rotate_cycles algorithm, 181
rotate_forward.annotated algorithm, 183
rotate_forward_nontrivial algorithm, 184
rotate_forward_step algorithm, 184
rotate_indexed_nontrivial algorithm, 181
rotate_nontrivial algorithm, 188
rotate_partial_nontrivial algorithm, 185
rotate_random_access_nontrivial algorithm,
  181
rotate_with_buffer_backward_nontrivial
  algorithm, 186
rotate_with_buffer_nontrivial algorithm, 185
rotation
  permutation, 178
  rearrangement, 179
S
schema, concept, 124
Schreier, Jozef, 55
Schwarz, Jerry, 150
segmented array, 221
segmented index, 221
select_0_2 algorithm, 53, 63
select_0_3 algorithm, 54
select_1_2 algorithm, 54
select_1_3 algorithm, 55
select_1_3_ab algorithm, 55
select_1_4 algorithm, 56, 59
select_1_4_ab algorithm, 56, 59
select_1_4_ab_cd algorithm, 56, 58
select_2_3 algorithm, 54
select_2_5 algorithm, 60
select_2_5_ab algorithm, 60
select_2_5_ab_cd algorithm, 59
semi (usage convention), 232
semigroup, 66
Semimodule concept, 69
Semiring concept, 68
semistable partition rearrangement, 192
sentinel, 101
Sequence concept, 216
  extent-based models, 219
  linked models, 219
set, 231
single-ended array, 220
single-extent array, 220
single-extent index, 221
single-pass traversal, 91
singly linked list, 218
sink, 149
size
  for array_k, 212
  for bounded_range, 214
  for Linearizable, 213
size of an orbit, 20
size of a range, 94
SizeType type function, 213
slanted index, 221
slow_quotient algorithm, 73
slow_remainder algorithm, 72
snapshot, 1
some algorithm, 97
sort_linked_nonempty_n algorithm, 142
sort_n algorithm, 207
sort_n_adaptive algorithm, 207
sort_n_with_buffer algorithm, 203
source, 90
space complexity, memory adaptive, 177
species
  abstract, 2
  concrete, 2
splicing link rearrangement, 219
split_copy algorithm, 158
split_linked algorithm, 137
stability, 52
  of merge, 203
  of partition, 192
  of sort, 204
  of sort on linked range, 142
stability index, 53
Standard Template Library, x
starting address, 4, 216
state of object, 4
Stein, Josef, 81
Stein gcd, 81
STL, x
store, 4
strengthened relation, 53
strengthening precondition, 38
strict property, 50
strict tail-recursive, 37
strictly increasing range, 103
strictly_increasing_counted_range property, 105
strictly_increasing_range property, 104
structural equality, 228
subpart of composite object, 216

subset (⊆), 231
subtraction, in additive group, 67
subtractive_gcd algorithm, 78
subtractive_gcd_nonzero algorithm, 77
successor
  definition space on range, 94
  of integer, 41
  of iterator, 91
sum (+)
  in additive semigroup, 66
  of integers, 18
  of iterator and integer, 92
swap algorithm, 224
swap_basic algorithm, 223
swap_ranges algorithm, 165
swap_ranges_bounded algorithm, 166
swap_ranges_n algorithm, 166
swap_step machine, 165
symmetric complement of a relation, 52
symmetric property, 50

T
tail-recursive form, 35
technique. See program transformation
  auxiliary computation during recursion, 176
  memory-adaptive algorithm, 177
  operation–accumulation procedure duality, 47
  reduction to constrained subproblem, 54
  returning useful information, 87, 96, 97,
    101–103, 106, 112, 152, 153, 159, 163,
    174, 179, 182, 211
  transformation–action duality, 28
  useful variations of an interface, 38
temporary_buffer type, 187
terminal element under transformation, 18
terminating algorithm, 23
three-valued compare, 63
Tighe, Joseph, 179
to-permutation, 172
total object state, 5
total procedure, 17
total value type, 2
TotallyOrdered concept, 62
total_ordering property, 51
trait class, 240
transformation, 17
  composing, 17, 32
cyclic element, 18
fixed point of, 170
identity, 170
into, 169
of program. See program transformation
one-to-one, 169
onto, 169
orbit, 18
power of \( f^n \), 17
terminal element, 18
Transformation concept, 17
transitive property, 49
transpose, operation algorithm, 201
transposition, 171
traversal
  multipass, 106
  single-pass, 91
  of tree, recursive, 119
transverse algorithm, 123
transverse_nonempty algorithm, 118
transverse_phased_rotating algorithm, 148
transverse_rotating algorithm, 146
transverse_step machine, 121
tree property, 117
tree_rotate machine, 145
trichotomy law, 51
triple type, 11
trivial cycle, 171
twice, 41
two-pointer header doubly linked list, 218
type
  array_k, 210
  bounded_range, 214
  computational basis, 6
  counter_machine, 200
  isomorphism, 86
  models concept, 11
  pair, 11, 209
  regular, 6
  temporary_buffer, 187
  triple, 11
  underlying_iterator, 225
  visit, 118
type attribute, 10
  Arity, 11
type concept, 11
type constructor, 11
type function, 11
Codomain, 11
DifferenceType, 113
DistanceType, 17, 91
Domain, 12
implemented via trait class, 240
InputType, 11
IteratorConcept, 187
IteratorType, 133, 134, 213
QuotientType, 72
SizeType, 213
UnderlyingType, 223
ValueType, 90, 149, 213
WeightType, 115
U
unambiguous value type, 3
UnaryFunction concept, 12
UnaryPredicate concept, 16
underlying type, 164, 223
  iterator adapters, 224
  proper, 223
UnderlyingType type function, 223
underlying_iterator type, 225
underlying_ref algorithm, 224
union (\( \cup \)), 231
uniquely represented object type, 5
uniquely represented value type, 2
univalent concept, 86
upper bound, 107
upper_bound_n algorithm, 109
upper_bound_predicate algorithm, 109
useful variations of an interface, 38
usefulness of concept, 87
V
value, 2
value type, 2
  ambiguous, 3
  properly partial, 2
  regular function on, 3
total, 2
  uniquely represented, 2
ValueType type function, 90, 149, 213
visit type, 118
W
weak (usage convention), 232
weak-trichotomy law, 51
<table>
<thead>
<tr>
<th>Term</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>weakening of concept</td>
<td>11</td>
</tr>
<tr>
<td>weak_ordering property</td>
<td>52</td>
</tr>
<tr>
<td>weak_range property</td>
<td>92</td>
</tr>
<tr>
<td>weight algorithm</td>
<td>122</td>
</tr>
<tr>
<td>WeightType type function</td>
<td>115</td>
</tr>
<tr>
<td>weight_recursive algorithm</td>
<td>117</td>
</tr>
<tr>
<td>weight_rotating algorithm</td>
<td>147</td>
</tr>
<tr>
<td>well-formed object</td>
<td>5</td>
</tr>
<tr>
<td>well-formed value</td>
<td>2</td>
</tr>
<tr>
<td>words in memory</td>
<td>4</td>
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<tr>
<td>writable range</td>
<td>150</td>
</tr>
<tr>
<td>writable_bounded_range property</td>
<td>150</td>
</tr>
<tr>
<td>writable_counted_range property</td>
<td>150</td>
</tr>
<tr>
<td>writable_weak_range property</td>
<td>150</td>
</tr>
<tr>
<td>write_aliased property</td>
<td>159</td>
</tr>
<tr>
<td>X</td>
<td></td>
</tr>
<tr>
<td>zero</td>
<td>41</td>
</tr>
</tbody>
</table>