Elements of Programming
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Elements of Programming

Alexander Stepanov
Paul McJones
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Contents

Preface ix
About the Authors xiii

1 Foundations 1
  1.1 Categories of Ideas: Entity, Species, Genus 1
  1.2 Values 2
  1.3 Objects 4
  1.4 Procedures 6
  1.5 Regular Types 6
  1.6 Regular Procedures 8
  1.7 Concepts 10
  1.8 Conclusions 14

2 Transformations and Their Orbits 15
  2.1 Transformations 15
  2.2 Orbits 18
  2.3 Collision Point 21
  2.4 Measuring Orbit Sizes 27
  2.5 Actions 28
  2.6 Conclusions 29

3 Associative Operations 31
  3.1 Associativity 31
  3.2 Computing Powers 33
3.3 Program Transformations 35
3.4 Special-Case Procedures 39
3.5 Parameterizing Algorithms 42
3.6 Linear Recurrences 43
3.7 Accumulation Procedures 46
3.8 Conclusions 47

4 Linear Orderings 49
4.1 Classification of Relations 49
4.2 Total and Weak Orderings 51
4.3 Order Selection 52
4.4 Natural Total Ordering 61
4.5 Clusters of Derived Procedures 62
4.6 Extending Order-Selection Procedures 63
4.7 Conclusions 63

5 Ordered Algebraic Structures 65
5.1 Basic Algebraic Structures 65
5.2 Ordered Algebraic Structures 70
5.3 Remainder 71
5.4 Greatest Common Divisor 76
5.5 Generalizing gcd 79
5.6 Stein gcd 81
5.7 Quotient 81
5.8 Quotient and Remainder for Negative Quantities 83
5.9 Concepts and Their Models 85
5.10 Computer Integer Types 87
5.11 Conclusions 88

6 Iterators 89
6.1 Readability 89
6.2 Iterators 90
6.3 Ranges 92
6.4 Readable Ranges 95
11 Partition and Merging 191
  11.1 Partition 191
  11.2 Balanced Reduction 198
  11.3 Merging 202
  11.4 Conclusions 208

12 Composite Objects 209
  12.1 Simple Composite Objects 209
  12.2 Dynamic Sequences 216
  12.3 Underlying Type 222
  12.4 Conclusions 225

Afterword 227

Appendix A Mathematical Notation 231

Appendix B Programming Language 233
  B.1 Language Definition 233
  B.2 Macros and Trait Structures 240

Bibliography 243

Index 247
This book applies the deductive method to programming by affiliating programs with the abstract mathematical theories that enable them to work. Specification of these theories, algorithms written in terms of these theories, and theorems and lemmas describing their properties are presented together. The implementation of the algorithms in a real programming language is central to the book. While the specifications, which are addressed to human beings, should, and even must, combine rigor with appropriate informality, the code, which is addressed to the computer, must be absolutely precise even while being general.

As with other areas of science and engineering, the appropriate foundation of programming is the deductive method. It facilitates the decomposition of complex systems into components with mathematically specified behavior. That, in turn, is a necessary precondition for designing efficient, reliable, secure, and economical software.

The book is addressed to those who want a deeper understanding of programming, whether they are full-time software developers, or scientists and engineers for whom programming is an important part of their professional activity.

The book is intended to be read from beginning to end. Only by reading the code, proving the lemmas, and doing the exercises can readers gain understanding of the material. In addition, we suggest several projects, some open-ended. While the book is terse, a careful reader will eventually see the connections between its parts and the reasons for our choice of material. Discovering the architectural principles of the book should be the reader’s goal.

We assume an ability to do elementary algebraic manipulations. We also assume familiarity with the basic vocabulary of logic and set theory at the level of undergraduate courses on discrete mathematics; Appendix A summarizes the notation that we use. We provide definitions of a few concepts of abstract algebra when they are

---

1. For a refresher on elementary algebra, we recommend Chrystal [1904].
needed to specify algorithms. We assume programming maturity and understanding of computer architecture\(^2\) and fundamental algorithms and data structures.\(^3\)

We chose C++ because it combines powerful abstraction facilities with faithful representation of the underlying machine.\(^4\) We use a small subset of the language and write requirements as structured comments. We hope that readers not already familiar with C++ are able to follow the book. Appendix B specifies the subset of the language used in the book.\(^5\) Wherever there is a difference between mathematical notation and C++, the typesetting and the context determine whether the mathematical or C++ meaning applies. While many concepts and programs in the book have parallels in STL (the C++ Standard Template Library), the book departs from some of the STL design decisions. The book also ignores issues that a real library, such as STL, has to address: namespaces, visibility, inline directives, and so on.

Chapter 1 describes values, objects, types, procedures, and concepts. Chapters 2–5 describe algorithms on algebraic structures, such as semigroups and totally ordered sets. Chapters 6–11 describe algorithms on abstractions of memory. Chapter 12 describes objects containing other objects. The Afterword presents our reflections on the approach presented by the book.

**Acknowledgments**

We are grateful to Adobe Systems and its management for supporting the Foundations of Programming course and this book, which grew out of it. In particular, Greg Gilley initiated the course and suggested writing the book; Dave Story and then Bill Hensler provided unwavering support. Finally, the book would not have been possible without Sean Parent’s enlightened management and continuous scrutiny of the code and the text. The ideas in the book stem from our close collaboration, spanning almost three decades, with Dave Musser. Bjarne Stroustrup deliberately evolved C++ to support these ideas. Both Dave and Bjarne were kind enough to come to San Jose and carefully review the preliminary draft. Sean Parent and Bjarne Stroustrup wrote the appendix defining the C++ subset used in the book. Jon Brandt reviewed multiple drafts of the book. John Wilkinson carefully read the final manuscript, providing innumerable valuable suggestions.

2. We recommend Patterson and Hennessy [2007].
3. For a selective but incisive introduction to algorithms and data structures, we recommend Tarjan [1983].
4. The standard reference is Stroustrup [2000].
5. The code in the book compiles and runs under Microsoft Visual C++ 9 and g++ 4. This code, together with a few trivial macros that enable it to compile, as well as unit tests, can be downloaded from www.elementsofprogramming.com.
The book has benefited significantly from the contributions of our editor, Peter Gordon, our project editor, Elizabeth Ryan, our copy editor, Evelyn Pyle, and the editorial reviewers: Matt Austern, Andrew Koenig, David Musser, Arch Robison, Jerry Schwarz, Jeremy Siek, and John Wilkinson.

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Finally, we are grateful to all the people who taught us through their writings or in person, and to the institutions that allowed us to deepen our understanding of programming.

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\(^6\) See www.elementsofprogramming.com for the up-to-date errata.
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About the Authors

Alexander Stepanov studied mathematics at Moscow State University from 1967 to 1972. He has been programming since 1972: first in the Soviet Union and, after emigrating in 1977, in the United States. He has programmed operating systems, programming tools, compilers, and libraries. His work on foundations of programming has been supported by GE, Brooklyn Polytechnic, AT&T, HP, SGI, and Adobe. In 1995 he received the *Dr. Dobb’s Journal* Excellence in Programming Award for the design of the C++ Standard Template Library.

Paul McJones studied engineering mathematics at the University of California, Berkeley, from 1967 to 1971. He has been programming since 1967 in the areas of operating systems, programming environments, transaction processing systems, and enterprise and consumer applications. He has been employed by the University of California, IBM, Xerox, Tandem, DEC, and Adobe. In 1982 he and his coauthors received the ACM Programming Systems and Languages Paper Award for their paper “The Recovery Manager of the System R Database Manager.”
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Chapter 2
Transformations and Their Orbits

This chapter defines a transformation as a unary regular function from a type to itself. Successive applications of a transformation starting from an initial value determine an orbit of this value. Depending only on the regularity of the transformation and the finiteness of the orbit, we implement an algorithm for determining orbit structures that can be used in different domains. For example, it could be used to detect a cycle in a linked list or to analyze a pseudorandom number generator. We derive an interface to the algorithm as a set of related procedures and definitions for their arguments and results. This analysis of an orbit-structure algorithm allows us to introduce our approach to programming in the simplest possible setting.

2.1 Transformations

While there are functions from any sequence of types to any type, particular classes of signatures commonly occur. In this book we frequently use two such classes: homogeneous predicates and operations. Homogeneous predicates are of the form $T \times \cdots \times T \rightarrow \text{bool}$; operations are functions of the form $T \times \cdots \times T \rightarrow T$. While there are $n$-ary predicates and $n$-ary operations, we encounter mostly unary and binary homogeneous predicates and unary and binary operations.

A predicate is a functional procedure returning a truth value:

\[
\text{Predicate}(P) \triangleq \\
\quad \text{FunctionalProcedure}(P) \\
\quad \land \text{Codomain}(P) = \text{bool}
\]
A homogeneous predicate is one that is also a homogeneous function:

\[
\text{HomogeneousPredicate}(P) \triangleq \\
\text{Predicate}(P) \\
\wedge \text{HomogeneousFunction}(P)
\]

A unary predicate is a predicate taking one parameter:

\[
\text{UnaryPredicate}(P) \triangleq \\
\text{Predicate}(P) \\
\wedge \text{UnaryFunction}(P)
\]

An operation is a homogeneous function whose codomain is equal to its domain:

\[
\text{Operation}(Op) \triangleq \\
\text{HomogeneousFunction}(Op) \\
\wedge \text{Codomain}(Op) = \text{Domain}(Op)
\]

Examples of operations:

```c
int abs(int x) {
    if (x < 0) return -x; else return x;
} // unary operation
```

```c
double euclidean_norm(double x, double y) {
    return sqrt(x * x + y * y);
} // binary operation
```

```c
double euclidean_norm(double x, double y, double z) {
    return sqrt(x * x + y * y + z * z);
} // ternary operation
```

**Lemma 2.1** \( \text{euclidean\_norm}(x, y, z) = \text{euclidean\_norm}(\text{euclidean\_norm}(x, y), z) \)

This lemma shows that the ternary version can be obtained from the binary version. For reasons of efficiency, expressiveness, and, possibly, accuracy, the ternary version is part of the computational basis for programs dealing with three-dimensional space.
A procedure is *partial* if its definition space is a subset of the direct product of the types of its inputs; it is *total* if its definition space is equal to the direct product. We follow standard mathematical usage, where partial function includes total function. We call partial procedures that are not total *nontotal*. Implementations of some total functions are nontotal on the computer because of the finiteness of the representation. For example, addition on signed 32-bit integers is nontotal.

A nontotal procedure is accompanied by a precondition specifying its definition space. To verify the correctness of a call of that procedure, we must determine that the arguments satisfy the precondition. Sometimes, a partial procedure is passed as a parameter to an algorithm that needs to determine at runtime the definition space of the procedural parameter. To deal with such cases, we define a definition-space *predicate* with the same inputs as the procedure; the predicate returns true if and only if the inputs are within the definition space of the procedure. Before a nontotal procedure is called, either its precondition must be satisfied, or the call must be guarded by a call of its definition-space predicate.

**Exercise 2.1** Implement a definition-space predicate for addition on 32-bit signed integers.

This chapter deals with unary operations, which we call *transformations*:

\[
\text{Transformation}(F) \triangleq \text{Operation}(F) \land \text{UnaryFunction}(F) \land \text{DistanceType} : \text{Transformation} \rightarrow \text{Integer}
\]

We discuss DistanceType in the next section.

Transformations are self-composable: \(f(x), f(f(x)), f(f(f(x)))\), and so on. The definition space of \(f(f(x))\) is the intersection of the definition space and result space of \(f\). This ability to self-compose, together with the ability to test for equality, allows us to define interesting algorithms.

When \(f\) is a transformation, we define its powers as follows:

\[
f^n(x) = \begin{cases} 
  x & \text{if } n = 0, \\
  f^{n-1}(f(x)) & \text{if } n > 0
\end{cases}
\]
Transformations and Their Orbits

To implement an algorithm to compute \( f^n(x) \), we need to specify the requirement for an integer type. We study various concepts describing integers in Chapter 5. For now we rely on the intuitive understanding of integers. Their models include signed and unsigned integral types, as well as arbitrary-precision integers, with these operations and literals:

<table>
<thead>
<tr>
<th>Specifications</th>
<th>C++</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sum</td>
<td>+</td>
</tr>
<tr>
<td>Difference</td>
<td>−</td>
</tr>
<tr>
<td>Product</td>
<td>*</td>
</tr>
<tr>
<td>Quotient</td>
<td>/</td>
</tr>
<tr>
<td>Remainder</td>
<td>mod</td>
</tr>
<tr>
<td>Zero</td>
<td>0</td>
</tr>
<tr>
<td>One</td>
<td>1</td>
</tr>
<tr>
<td>Two</td>
<td>2</td>
</tr>
</tbody>
</table>

where \( I \) is an integer type.

That leads to the following algorithm:

```cpp
template<typename F, typename N>
requires(Transformation(F) && Integer(N))
Domain(F) power_unary(Domain(F) x, N n, F f)
{
    // Precondition: \( n \geq 0 \land (\forall i \in N) 0 < i \leq n \Rightarrow f^i(x) \) is defined
    while (n != N(0)) {
        n = n - N(1);
        x = f(x);
    }
    return x;
}
```

2.2 Orbits

To understand the global behavior of a transformation, we examine the structure of its orbits: elements reachable from a starting element by repeated applications of the transformation. \( y \) is reachable from \( x \) under a transformation \( f \) if for some \( n \geq 0, y = f^n(x) \). \( x \) is cyclic under \( f \) if for some \( n \geq 1, x = f^n(x) \). \( x \) is terminal under \( f \) if and only if \( x \) is not in the definition space of \( f \). The orbit of \( x \) under a transformation \( f \) is the set of all elements reachable from \( x \) under \( f \).
Lemma 2.2  An orbit does not contain both a cyclic and a terminal element.

Lemma 2.3  An orbit contains at most one terminal element.

If \( y \) is reachable from \( x \) under \( f \), the distance from \( x \) to \( y \) is the least number of transformation steps from \( x \) to \( y \). Obviously, distance is not always defined.

Given a transformation type \( F \), DistanceType\((F)\) is an integer type large enough to encode the maximum number of steps by any transformation \( f \in F \) from one element of \( T = \text{Domain}(F) \) to another. If type \( T \) occupies \( k \) bits, there can be as many as \( 2^k \) values but only \( 2^k - 1 \) steps between distinct values. Thus if \( T \) is a fixed-size type, an integral type of the same size is a valid distance type for any transformation on \( T \). (Instead of using the distance type, we allow the use of any integer type in \( \text{power\_unary} \), since the extra generality does not appear to hurt there.) It is often the case that all transformation types over a domain have the same distance type. In this case the type function DistanceType is defined for the domain type and defines the corresponding type function for the transformation types.

The existence of DistanceType leads to the following procedure:

```cpp
template<typename F>
  requires(Transformation(F))
DistanceType(F) distance(Domain(F) x, Domain(F) y, F f)
{
  // Precondition: \( y \) is reachable from \( x \) under \( f \)
  typedef DistanceType(F) N;
  N n(0);
  while (x != y) {
    x = f(x);
    n = n + N(1);
  }
  return n;
}
```

Orbits have different shapes. An orbit of \( x \) under a transformation is

- **infinite** if it has no cyclic or terminal elements
- **terminating** if it has a terminal element
- **circular** if \( x \) is cyclic
- **\( \rho \)-shaped** if \( x \) is not cyclic, but its orbit contains a cyclic element

An orbit of \( x \) is **finite** if it is not infinite. Figure 2.1 illustrates the various cases.
The *orbit cycle* is the set of cyclic elements in the orbit and is empty for infinite and terminating orbits. The *orbit handle*, the complement of the orbit cycle with respect to the orbit, is empty for a circular orbit. The *connection point* is the first cyclic element, and is the first element of a circular orbit and the first element after the handle for a $\rho$-shaped orbit. The *orbit size* $o$ of an orbit is the number of distinct elements in it. The *handle size* $h$ of an orbit is the number of elements in the orbit handle. The *cycle size* $c$ of an orbit is the number of elements in the orbit cycle.

**Lemma 2.4** $o = h + c$

**Lemma 2.5** The distance from any point in an orbit to a point in a cycle of that orbit is always defined.

**Lemma 2.6** If $x$ and $y$ are distinct points in a cycle of size $c$,

$$c = \text{distance}(x, y, f) + \text{distance}(y, x, f)$$

**Lemma 2.7** If $x$ and $y$ are points in a cycle of size $c$, the distance from $x$ to $y$ satisfies

$$0 \leq \text{distance}(x, y, f) < c$$
2.3 Collision Point

If we observe the behavior of a transformation, without access to its definition, we cannot determine whether a particular orbit is infinite: It might terminate or cycle back at any point. If we know that an orbit is finite, we can use an algorithm to determine the shape of the orbit. Therefore there is an implicit precondition of orbit finiteness for all the algorithms in this chapter.

There is, of course, a naive algorithm that stores every element visited and checks at every step whether the new element has been previously encountered. Even if we could use hashing to speed up the search, such an algorithm still would require linear storage and would not be practical in many applications. However, there is an algorithm that requires only a constant amount of storage.

The following analogy helps to understand the algorithm. If a fast car and a slow one start along a path, the fast one will catch up with the slow one if and only if there is a cycle. If there is no cycle, the fast one will reach the end of the path before the slow one. If there is a cycle, by the time the slow one enters the cycle, the fast one will already be there and will catch up eventually. Carrying our intuition from the continuous domain to the discrete domain requires care to avoid the fast one skipping past the slow one.¹

The discrete version of the algorithm is based on looking for a point where fast meets slow. The collision point of a transformation \( f \) and a starting point \( x \) is the unique \( y \) such that

\[
y = f^n(x) = f^{2n+1}(x)
\]

and \( n \geq 0 \) is the smallest integer satisfying this condition. This definition leads to an algorithm for determining the orbit structure that needs one comparison of fast and slow per iteration. To handle partial transformations, we pass a definition-space predicate to the algorithm:

\[
\text{template<typename F, typename P>}
\]
\[
\text{requires(Transformation(F) && UnaryPredicate(P) &&}
\]
\[
\text{Domain(F) == Domain(P))}
\]
\[
\text{Domain(F) collision_point(const Domain(F)& x, F f, P p)}
\]
\[
\{
\text{ // Precondition: p(x) ⇔ f(x) is defined}
\text{ if (!p(x)) return x;}
\]

Domain(F) slow = x;   // slow = f^0(x)
Domain(F) fast = f(x);   // fast = f^1(x)
   // n ← 0 (completed iterations)
while (fast != slow) {
   slow = f(slow);   // slow = f^n(x) ∧ fast = f^{2n+1}(x)
   fast = f(fast);   // slow = f^{n+1}(x) ∧ fast = f^{2n+2}(x)
   if (!p(fast)) return fast;
   fast = f(fast);   // slow = f^{n+1}(x) ∧ fast = f^{2n+3}(x)
   if (!p(fast)) return fast;
   fast = f(fast);   // n ← n + 1
}
return fast;   // slow = f^n(x) ∧ fast = f^{2n+1}(x)
   // Postcondition: return value is terminal point or collision point

We establish the correctness of collision_point in three stages: (1) verifying that it never applies f to an argument outside the definition space; (2) verifying that if it terminates, the postcondition is satisfied; and (3) verifying that it always terminates.

While f is a partial function, its use by the procedure is well defined, since the movement of fast is guarded by a call of p. The movement of slow is unguarded, because by the regularity of f, slow traverses the same orbit as fast, so f is always defined when applied to slow.

The annotations show that if, after n ≥ 0 iterations, fast becomes equal to slow, then fast = f^{2n+1}(x) and slow = f^n(x). Moreover, n is the smallest such integer, since we checked the condition for every i < n.

If there is no cycle, p will eventually return false because of finiteness. If there is a cycle, slow will eventually reach the connection point (the first element in the cycle). Consider the distance d from fast to slow at the top of the loop when slow first enters the cycle: 0 ≤ d < c. If d = 0, the procedure terminates. Otherwise the distance from fast to slow decreases by 1 on each iteration. Therefore the procedure always terminates; when it terminates, slow has moved a total of h + d steps.

The following procedure determines whether an orbit is terminating:

template<typename F, typename P>
   requires(Transformation(F) && UnaryPredicate(P) &&
            Domain(F) == Domain(P))
bool terminating(const Domain(F)& x, F f, P p)
2.3 Collision Point

```cpp
{
    // Precondition: p(x) ⇔ f(x) is defined
    return !p(collision_point(x, f, p));
}

Sometimes we know either that the transformation is total or that the orbit is nonterminating for a particular starting element. For these situations it is useful to have a specialized version of collision_point:

```cpp
template<typename F>
requires(Transformation(F))
Domain(F)
collision_point_nonterminating_orbit(const Domain(F)& x, F f)
{
    Domain(F) slow = x;  // slow = f^0(x)
    Domain(F) fast = f(x);  // fast = f^1(x)
    // n ← 0 (completed iterations)
    while (fast != slow) {
        slow = f(slow);  // slow = f^n(x) ∧ fast = f^{2n+1}(x)
        fast = f(fast);  // slow = f^{n+1}(x) ∧ fast = f^{2n+1}(x)
    }
    return fast;  // slow = f^n(x) ∧ fast = f^{2n+1}(x)
    // Postcondition: return value is collision point
}
```

In order to determine the cycle structure—handle size, connection point, and cycle size—we need to analyze the position of the collision point.

When the procedure returns the collision point

\[ f^n(x) = f^{2n+1}(x) \]

\( n \) is the number of steps taken by slow, and \( 2n + 1 \) is the number of steps taken by fast.

\[ n = h + d \]
where $h$ is the handle size and $0 \leq d < c$ is the number of steps taken by slow inside the cycle. The number of steps taken by fast is

$$2n + 1 = h + d + qc$$

where $q > 0$ is the number of full cycles completed by fast when it collides with slow. Since $n = h + d$,

$$2(h + d) + 1 = h + d + qc$$

Simplifying gives

$$qc = h + d + 1$$

Let us represent $h$ modulo $c$:

$$h = mc + r$$

with $0 \leq r < c$. Substitution gives

$$qc = mc + r + d + 1$$

or

$$d = (q - m)c - r - 1$$

$0 \leq d < c$ implies

$$q - m = 1$$

so

$$d = c - r - 1$$

and $r + 1$ steps are needed to complete the cycle.

Therefore the distance from the collision point to the connection point is

$$e = r + 1$$

In the case of a circular orbit $h = 0$, $r = 0$, and the distance from the collision point to the beginning of the orbit is

$$e = 1$$
2.3 Collision Point

Circularity, therefore, can be checked with the following procedures:

```cpp
template<typename F>
    requires(Transformation(F))
bool circular_nonterminating_orbit(const Domain(F)& x, F f) {
    return x == f(collision_point_nonterminating_orbit(x, f));
}

template<typename F, typename P>
    requires(Transformation(F) && UnaryPredicate(P) && Domain(F) == Domain(P))
bool circular(const Domain(F)& x, F f, P p) {
    // Precondition: p(x) ⇔ f(x) is defined
    Domain(F) y = collision_point(x, f, p);
    return p(y) && x == f(y);
}
```

We still don’t know the handle size \( h \) and the cycle size \( c \). Determining the latter is simple once the collision point is known: Traverse the cycle and count the steps.

To see how to determine \( h \), let us look at the position of the collision point:

\[
\begin{align*}
  f^{h+d}(x) &= f^{h+c-r-1}(x) = f^{mc+r+c-r-1}(x) = f^{m+1}c-1(x)
\end{align*}
\]

Taking \( h + 1 \) steps from the collision point gets us to the point \( f^{m+1}c+h(x) \), which equals \( f^h(x) \), since \((m + 1)c\) corresponds to going around the cycle \( m + 1 \) times. If we simultaneously take \( h \) steps from \( x \) and \( h + 1 \) steps from the collision point, we meet at the connection point. In other words, the orbits of \( x \) and 1 step past the collision point converge in exactly \( h \) steps, which leads to the following sequence of algorithms:

```cpp
template<typename F>
    requires(Transformation(F))
Domain(F) convergent_point(Domain(F) x0, Domain(F) x1, F f) {
    // Precondition: (\exists n \in DistanceType(F)) n \geq 0 \land f^n(x0) = f^n(x1)
    while (x0 != x1) {
    ```
transformations and their orbits

\[ x_0 = f(x_0); \]
\[ x_1 = f(x_1); \]
}
return x0;
}

template<typename F>
    requires(Transformation(F))
Domain(F)
connection_point_nonterminating_orbit(const Domain(F)& x, F f)
{
    return convergent_point(
        x,
        f(collision_point_nonterminating_orbit(x, f)),
        f);
}

template<typename F, typename P>
    requires(Transformation(F) && UnaryPredicate(P) &&
        Domain(F) == Domain(P))
Domain(F) connection_point(const Domain(F)& x, F f, P p)
{
    // Precondition: p(x) ⇔ f(x) is defined
    Domain(F) y = collision_point(x, f, p);
    if (!p(y)) return y;
    return convergent_point(x, f(y), f);
}

Lemma 2.8 If the orbits of two elements intersect, they have the same cyclic elements.

Exercise 2.2 Design an algorithm that determines, given a transformation and its definition-space predicate, whether the orbits of two elements intersect.

Exercise 2.3 The precondition of convergent_point ensures termination. Implement an algorithm convergent_point_guarded for use when that precondition is not known to hold, but there is an element in common to the orbits of both x0 and x1.
2.4 Measuring Orbit Sizes

The natural type to use for the sizes $o$, $h$, and $c$ of an orbit on type $T$ would be an integer count type large enough to count all the distinct values of type $T$. If a type $T$ occupies $k$ bits, there can be as many as $2^k$ values, so a count type occupying $k$ bits could not represent all the counts from 0 to $2^k$. There is a way to represent these sizes by using distance type.

An orbit could potentially contain all values of a type, in which case $o$ might not fit in the distance type. Depending on the shape of such an orbit, $h$ and $c$ would not fit either. However, for a $\rho$-shaped orbit, both $h$ and $c$ fit. In all cases each of these fits: $o - 1$ (the maximum distance in the orbit), $h - 1$ (the maximum distance in the handle), and $c - 1$ (the maximum distance in the cycle). That allows us to implement procedures returning a triple representing the complete structure of an orbit, where the members of the triple are as follows:

<table>
<thead>
<tr>
<th>Case</th>
<th>$m_0$</th>
<th>$m_1$</th>
<th>$m_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Terminating</td>
<td>$h - 1$</td>
<td>0</td>
<td>terminal element</td>
</tr>
<tr>
<td>Circular</td>
<td>0</td>
<td>$c - 1$</td>
<td>$x$</td>
</tr>
<tr>
<td>$\rho$-shaped</td>
<td>$h$</td>
<td>$c - 1$</td>
<td>connection point</td>
</tr>
</tbody>
</table>

```cpp
template<typename F>
requires(Transformation(F))
triple<DistanceType(F), DistanceType(F), Domain(F)>
orbit_structure_nonterminating_orbit(const Domain(F)& x, F f)
{
    typedef DistanceType(F) N;
    Domain(F) y = connection_point_nonterminating_orbit(x, f);
    return triple<N, N, Domain(F)>(distance(x, y, f),
                                     distance(f(y), y, f),
                                     y);
}

template<typename F, typename P>
requires(Transformation(F) &&
          UnaryPredicate(P) && Domain(F) == Domain(P))
triple<DistanceType(F), DistanceType(F), Domain(F)>
orbit_structure(const Domain(F)& x, F f, P p)
{
    // Precondition: p(x) ⇔ f(x) is defined
```
typedef DistanceType(F) N;
Domain(F) y = connection_point(x, f, p);
N m = distance(x, y, f);
N n(0);
if (p(y)) n = distance(f(y), y, f);
// Terminating: m = h − 1 ∧ n = 0
// Otherwise: m = h ∧ n = c − 1
return triple<N, N, Domain(F)>(m, n, y);
}

Exercise 2.4 Derive formulas for the count of different operations (f, p, equality) for the algorithms in this chapter.

Exercise 2.5 Use orbit_structure_nonterminating_orbit to determine the average handle size and cycle size of the pseudorandom number generators on your platform for various seeds.

2.5 Actions

Algorithms often use a transformation f in a statement like

\[ x = f(x); \]

Changing the state of an object by applying a transformation to it defines an action on the object. There is a duality between transformations and the corresponding actions: An action is definable in terms of a transformation, and vice versa:

\begin{verbatim}
void a(T& x) { x = f(x); }  // action from transformation
\end{verbatim}

and

\begin{verbatim}
T f(T x) { a(x); return x; }  // transformation from action
\end{verbatim}

Despite this duality, independent implementations are sometimes more efficient, in which case both action and transformation need to be provided. For example, if a transformation is defined on a large object and modifies only part of its overall state, the action could be considerably faster.

Exercise 2.6 Rewrite all the algorithms in this chapter in terms of actions.
Project 2.1 Another way to detect a cycle is to repeatedly test a single advancing element for equality with a stored element while replacing the stored element at ever-increasing intervals. This and other ideas are described in Sedgewick, et al. [1979], Brent [1980], and Levy [1982]. Implement other algorithms for orbit analysis, compare their performance for different applications, and develop a set of recommendations for selecting the appropriate algorithm.

2.6 Conclusions

Abstraction allowed us to define abstract procedures that can be used in different domains. Regularity of types and functions is essential to make the algorithms work: fast and slow follow the same orbit because of regularity. Developing nomenclature is essential (e.g., orbit kinds and sizes). Affiliated types, such as distance type, need to be precisely defined.
Index

→ (function), 231
− (additive inverse), in additive group, 67
∧ (and), 231
− (difference)
in additive group, 67
in cancellable monoid, 72
of integers, 18
of iterator and integer, 111
of iterators, 93
× (direct product), 231
∈ (element), 231
= (equality), 7
for array_k, 212
for pair, 210
≜ (equals by definition), 12, 231
⇔ (equivalent), 231
∃ (exists), 231
∀ (for all), 231
> (greater), 62
≥ (greater or equal), 62
⇒ (implies), 231
[] (index)
for array_k, 211
for bounded_range, 214
≠ (inequality), 7, 62
∩ (intersection), 231
< (less), 62
for array_k, 212
natural total ordering, 61
for pair, 210
≤ (less or equal), 62
† (maps to), 231
¬ (not), 231
∨ (or), 231
α^n (power of associative operation), 32
f^n (power of transformation), 17
≺ (precedes), 95
≤ (precedes or equal), 95
· (product)
of integers, 18
in multiplicative semigroup, 66
in semimodule, 69
/ (quotient), of integers, 18
[f, l] (range, closed bounded), 94
[[f, n]] (range, closed weak or counted), 94
[f, l) (range, half-open bounded), 94
[[f, n) (range, half-open weak or counted), 94
⊂ (subset), 231
+ (sum)
in additive semigroup, 66
of integers, 18
of iterator and integer, 92
∪ (union), 231

A
abs algorithm, 16, 71
absolute value, properties, 71
abstract entity, 1
abstract genus, 2
abstract procedure, 13
overloading, 43
abstract species, 2
accumulation procedure, 46
accumulation variable
elimination, 39
introduction, 35
action, 28
acyclic descendants of bifurcate coordinate, 116
additive inverse (−), in additive group, 67
AdditiveGroup concept, 67
AdditiveMonoid concept, 67
AdditiveSemigroup concept, 66
address, 4
    abstracted by iterator, 89
add_to_counter algorithm, 199
advance_tail machine, 135
algorithm. See machine
    abs, 16, 71
    add_to_counter, 199
    all, 97
    bifurcate_compare, 131
    bifurcate_compare_nonempty, 130
    bifurcate_equivalent, 129
    bifurcate_equivalent_nonempty, 128
    bifurcate_isomorphic, 126
    bifurcate_isomorphic_nonempty, 125
circular, 25
circular_nonterminating_orbit, 25
collision_point, 22
collision_point_nonterminating_orbit, 23
    combine_copy, 160
    combine_copy_backward, 162
    combine_linked_nonempty, 138
    combine_ranges, 196
    compare_strict_or_reflexive, 57–58
complement, 50
    complement_of_converse, 50
    connection_point, 26
    connection_point_nonterminating_orbit, 26
convergent_point, 26
corver, 50
copy, 152
copy_backward, 155
copy_bounded, 153
copy_if, 158
copy_n, 154
copy_select, 158
count_if, 97, 98
cycle_from, 173
cycle_to, 173
distance, 19
euclidean_norm, 16
exchange_values, 164
fast_subtractive_gcd, 78
fibonacci, 46
find, 96
    find_adjacent_mismatch, 103
    find_adjacent_mismatch_forward, 106, 135
    find_backward_if, 112
    find_if, 97
find_if_not_unguarded, 102
find_if_unguarded, 101
find_last, 136
find_mismatch, 102
find_n, 101
find_not, 97
for_each, 96
    for_each_n, 101
gcd, 80
height, 122
height_recursive, 118
increment, 91
is_left_successor, 119
is_right_successor, 120
k_rotate_from_permutation_indexed, 180
k_rotate_from_permutation_random_access, 180
largest_doubling, 75
lexicographical_compare, 129
    lexicographical_equal, 127
    lexicographical_equivalent, 127
    lexicographical_less, 130
lower_bound_n, 109
lower_bound_predicate, 108
median_5, 61
memory-adaptive, 177
merge_copy, 163
    merge_copy_backward, 163
    merge_linked_nonempty, 141
    merge_n_adaptive, 206
    merge_n_with_buffer, 202
none, 97
not_all, 97
orbit_structure, 28
    orbit_structure_nonterminating_orbit, 27
partitioned_at_point, 191
partition_bidirectional, 194
    partition_copy, 160
    partition_copy_n, 160
    partition_linked, 140
    partition_point, 107
    partition_point_n, 107
    partition_semistable, 192
    partition_single_cycle, 194
    partition_stable_iterative, 201
    partition_stable_n, 197
    partition_stable_n_adaptive, 197
    partition_stable_n_nonempty, 197
algorithm. See machine (cont.)
partition_stable_singleton, 196
partition_stable_with_buffer, 195
partition_trivial, 198
phased_applicator, 147
potential_partition_point, 191
power, 42
power_accumulate, 41
power_accumulate_positive, 41
power_left_associated vs. power_0, 34
power_right_associated, 33
power unary, 18
predicate_source, 140
quotient_remainder, 85
quotient_remainder_nonnegative, 82
quotient_remainder_nonnegative_iterative, 83
reachable, 121
reduce, 99
reduce_balanced, 200
reduce_nonempty, 99
reduce_nonzeroes, 100
relation_source, 141
remainder, 84
remainder_nonnegative, 74
remainder_nonnegative_iterative, 75
reverse_append, 139, 140
reverse_bidirectional, 175
reverse_copy, 156
reverse_copy_backward, 156
reverse_indexed, 186
reverse_n_adaptive, 178
reverse_n_bidirectional, 175
reverse_n_forward, 177
reverse_n_indexed, 175
reverse_n_with_buffer, 176
reverse_swap_ranges, 167
reverse_swap_ranges_bounded, 167
reverse_swap_ranges_n, 168
reverse_with_temporary_buffer, 187, 225
rotate, 187
rotate_bidirectional_nontrivial, 182
rotate_cycles, 181
rotate_forward_annotated, 183
rotate_forward_nontrivial, 184
rotate_forward_step, 184
rotate_indexed_nontrivial, 181
rotate_nontrivial, 188
rotate_partial_nontrivial, 185
rotate_random_access_nontrivial, 181
rotate_with_buffer_backward_nontrivial, 186
rotate_with_buffer_nontrivial, 185
select.0.2, 53, 63
select.0.3, 54
select.1.2, 54
select.1.3, 55
select.1.3_ab, 55
select.1.4, 56, 59
select.1.4_ab, 56, 59
select.1.4_ab_cd, 56, 58
select.2.3, 54
select.2.5, 60
select.2.5_ab, 60
select.2.5_ab_cd, 59
slow_quotient, 73
slow_remainder, 72
some, 97
sort_linked_nonempty_n, 142
sort_n, 207
sort_n_adaptive, 207
sort_n_with_buffer, 203
split_copy, 158
split_linked, 137
subtractive_gcd, 78
subtractive_gcd_nonzero, 77
swap, 224
swap_basic, 223
swap_ranges, 165
swap_ranges_bounded, 166
swap_ranges_n, 166
terminating, 23
transpose_operation, 201
traverse, 123
traverse_nonempty, 118
traverse_phased_rotating, 148
traverse_rotating, 146
underlying_ref, 224
upper_bound_n, 109
upper_bound_predicate, 109
weight, 122
weight_recursive, 117
weight_rotating, 147
aliased property, 150
aliased write-read, 150
aliased write-write, 159
all algorithm, 97
ambiguous value type, 3
amortized complexity, 219
and ($\land$), 231
annihilation property, 68
annotation variable, 183
ArchimedeanGroup concept, 83
ArchimedeanMonoid concept, 72
area of object, 227
Aristotle, 77
Arity type attribute, 11
array, varieties, 220–221
array type, 210
Artin, Emil, 13
assignment, 7
for array_k, 211
for pair, 210
associative operation, 31, 98
power of ($a^n$), 32
associative property, 31
exploited by power, 33
partially_associative, 98
of permutation composition, 170
asymmetric property, 50
attribute, 1
auxiliary computation during recursion, 176
Axiom of Archimedes, 72, 73

B
backward movement in range, 112
BackwardLinker concept, 134
backward_offset property, 161
basic singly linked list, 218
begin
for array_k, 211
for bounded_range, 214
for Linearizable, 213
behavioral equality, 3, 228
BidirectionalBifurcateCoordinate concept, 119–120
BidirectionalIterator concept, 111
BidirectionalLinker concept, 134
BifurcateCoordinate concept, 115
bifurcate_compare algorithm, 131
bifurcate_compare_nonempty algorithm, 130
bifurcate_equivalent algorithm, 129
bifurcate_equivalent_nonempty algorithm, 128
bifurcate_isomorphic algorithm, 126
bifurcate_isomorphic_nonempty algorithm, 125
BinaryOperation concept, 31
binary_scale_down_nonnegative, 40
binary_scale_up_nonnegative, 40
bisection technique, 107
Bolzano, Bernard, 107
bounded integer type, 87
bounded range, 93
bounded_range property, 93
bounded_range type, 214
Brandt, Jon, 193

C
CancellableMonoid concept, 72
cancellation in monoid, 72
categories of ideas, 1
Cauchy, Augustin Louis, 107
circular algorithm, 25
circular array, 220
circular doubly linked list, 218
circular singly linked list, 218
circular_nonterminating_orbit algorithm, 25
closed bounded range ([f, l]), 94
closed interval, 231
closed weak or counted range ([f, n]), 94
clusters of derived procedures, 62
codomain, 10
Codomain type function, 11
Collins, George, 13
collision point of orbit, 21
collision_point algorithm, 22
collision_point_nonterminating_orbit algorithm, 23
combine_copy algorithm, 160
combine_copy_backward algorithm, 162
combine_linked_nonempty algorithm, 138
combine_ranges algorithm, 196
common-subexpression elimination, 35
commutative property, 66
CommutativeRing concept, 69
CommutativeSemiring concept, 68
compare_strict_or_reflexive algorithm, 57–58
complement algorithm, 50
complement of converse of relation, 50
complement of relation, 50
complement_of_converse algorithm, 50
complement_of_converse property, 104
complexity
amortized, 219
of empty, 213
of indexing of a sequence, 213
of regular operations, 227
of source, 90
of successor, 92
composite object, 215
composition
of permutations, 170
of transformations, 17, 32
computational basis, 6
concept, 11
AdditiveGroup, 67
AdditiveMonoid, 67
AdditiveSemigroup, 66
ArchimedeanGroup, 83
ArchimedeanMonoid, 72
BackwardLinker, 134
BidirectionalBifurcateCoordinate, 119–120
BidirectionalIterator, 111
BidirectionalLinker, 134
BifurcateCoordinate, 115
BinaryOperation, 31
CancellableMonoid, 72
CommutativeRing, 69
CommutativeSemiring, 68
consistent, 87
DiscreteArchimedeanRing, 86
DiscreteArchimedeanSemiring, 85
EmptyLinkedBifurcateCoordinate, 144
EuclideanMonoid, 77
EuclideanSemimodule, 80
EuclideanSemiring, 79
examples from C++ and STL, 11
ForwardIterator, 106
ForwardLinker, 133
FunctionalProcedure, 11
HalvableMonoid, 74
HomogeneousFunction, 12
HomogeneousPredicate, 16
IndexedIterator, 110
Integer, 18, 40
Iterator, 91
Linearizable, 213
LinkedBifurcateCoordinate, 144
modeled by type, 11
Module, 70
MultiplicativeGroup, 68
MultiplicativeMonoid, 67
MultiplicativeSemigroup, 66
NonnegativeDiscreteArchimedeanSemiring, 86
Operation, 16
OrderedAdditiveGroup, 70
OrderedAdditiveMonoid, 70
OrderedAdditiveSemigroup, 70
Predicate, 15
RandomAccessIterator, 113
refinement, 11
Regular, 11
Relation, 49
relational concept, 69
Ring, 69
Semimodule, 69
Semiring, 68
Sequence, 216
TotallyOrdered, 62
Transformation, 17
type concept, 11
UnaryFunction, 12
UnaryPredicate, 16
univalent, 86
useful, 87
weakening, 11
concept dispatch, 106, 187
concept schema
composite object, 216
coordinate structure, 124
concept tag type, 187
concrete entity, 1
concrete genus, 2
concrete species, 2
connectedness of composite object, 215
connection point of orbit, 20
connection_point algorithm, 26
connection_point_nonterminating_orbit algorithm, 26
connectors, 229
consistency of concept’s axioms, 87
constant-size sequence, 216
constructor, 7
container, 213
convergent_point algorithm, 26
converse algorithm, 50
converse of relation, 50
coordinate structure
  bifurcate coordinate, 115
  of composite object, 215
  concept schema, 124
  iterator, 89
copy algorithm, 152
copy constructor, 8
  for array_k, 211
  for pair, 210
copy of object, 5
copying rearrangement, 172
copy_backward algorithm, 155
copy_backward_step machine, 154
copy_bounded algorithm, 153
copy_if algorithm, 158
copy_n algorithm, 154
copy_select algorithm, 158
copy_step machine, 152
counted_range property, 93
counter_machine type, 200
count_down machine, 153
count_if algorithm, 97, 98
cycle detection intuition, 21
cycle in a permutation, 171
cycle of orbit, 20
cycle size, 20
cycle_from algorithm, 173
cycle_to algorithm, 173
cyclical element under transformation, 18
cyclical permutation, 171

D
DAG (directed acyclic graph), 116
datum, 2
de Bruijn, N. G., 74
default constructor, 8
  for array_k, 211
  for pair, 209
default ordering, 62
default total ordering, 62
  importance of, 228
definition space, 9
definition-space predicate, 17
dependence of axiom, 86
deref, 150
derived relation, 50

descendant of bifurcate coordinate, 116
destructor, 7
  for pair, 210
difference (−)
  in additive group, 67
  in cancellable monoid, 72
  of integers, 18
  of iterator and integer, 111
  of iterators, 93
difference type function, 113
direct product (×), 231
directed acyclic graph, 116
DiscreteArchimedeanRing concept, 86
DiscreteArchimedeanSemiring concept, 85
discreteness property, 85
disjoint property, 134
disjointness of composite object, 216
distance algorithm, 19
distance in orbit, 19
distance type function, 17, 91
distributive property, holds for semiring, 68
divisibility on an Archimedean monoid, 76
division, 68
domain, 10
Domain type function, 12
double-ended array, 220
doubly linked list, 218–219
Dudziński, Krzysztof, 206
dummy node doubly linked list, 218
Dydek, Andrzej, 206
dynamic-size sequence, 216
E
efficient computational basis, 6
element (∈), 231
eliminating common subexpression, 35
empty
  for array_k, 212
  for bounded_range, 214
  for Linearizable, 213
empty coordinate, 144
empty range, 95
EmptyLinkedBifurcateCoordinate concept, 144
end
for array_k, 211
for bounded_range, 214
for Linearizable, 213
entity, 1
equality
=, 7
≠, 62
for array_k, 212
behavioral, 3, 228
equal for Regular, 127
for objects, 5
for pair, 210
for regular type, 7
representational, 3, 228
structural, 228
for uniquely represented type, 3
for value type, 3
equals by definition (≜), 12, 231
equational reasoning:, 4
equivalence class, 51
equivalence property, 51
equivalent (⇔), 231
equivalent coordinate collections, 126
erasure in a sequence, 217
Euclidean function, 79
EuclideanMonoid concept, 77
EuclideanSemimodule concept, 80
EuclideanSemiring concept, 79
euclidean_norm algorithm, 16
even, 40
exchange_values algorithm, 164
exists (∃), 231
expressive computational basis, 6

F
fast_subtractive_gcd algorithm, 78
fibonacci algorithm, 46
Fibonacci sequence, 45
find algorithm, 96
find_adjacent_mismatch algorithm, 103
find_adjacent_mismatch_forward algorithm, 106, 135
find_backward_if algorithm, 112
find_if algorithm, 97
find_if_not, 97
find_if_not_unguarded algorithm, 102
find_if_unguarded algorithm, 101
find_last algorithm, 136
find_mismatch algorithm, 102
find_n algorithm, 101
find_not algorithm, 97
finite order, under associative operation, 32
finite set, 171
first-last singly linked list, 218
fixed point of transformation, 170
fixed-size range, 216
Floyd, Robert W., 21
for all (∀), 231
ForwardIterator concept, 106
ForwardLinker concept, 133
forward_offset property, 162
for_each algorithm, 96
for_each_n algorithm, 101
Frobenius, Georg Ferdinand, 32
from_permutation, 172
function, 2
→, 231
on abstract entities, 2
on values, 3
function object, 9, 96, 236
functional procedure, 9
FunctionalProcedure concept, 11

G
garbage collection, 230
Gaussian integers, 40
Stein's algorithm, 81
gcd, 76
Stein, 81
subtractive, 76
gcd algorithm, 80
genus, 2
global state, 6
goto statement, 148
greater (>), 62
greater or equal (≥), 62
greatest common divisor (gcd), 76
group, 67
do of permutations, 170

H
half_nonnegative, 40
half-open bounded range ([f, l]), 94
half-open interval, 231
half-open weak or counted range ([f, n]), 94
HalvableMonoid concept, 74
handle of orbit, 20
handle size, 20
header of composite object, 217
height algorithm, 122
height of bifurcate coordinate (DAG), 116
height_recursive algorithm, 118
Ho, Wilson, 182
Hoare, C. A. R., 195
homogeneous functional procedure, 10
HomogeneousFunction concept, 12
HomogeneousPredicate concept, 16
I
ideas, categories of, 1
identity
  of concrete entity, 1
  of object, 5
identity element, 65
identity token, 5
identity transformation, 170
identity_element property, 65
implies (⇒), 231
inconsistency of concept, 87
increasing range, 103
increasing_counted_range property, 105
increasing_range property, 105
increment algorithm, 91
independence of proposition, 86
index ([ ])
  for array_k, 211
  for bounded_range, 214
index permutation, 172
index of segmented array, 221
indexed iterator
  equivalent to random-access iterator, 113
IndexedIterator concept, 110
inequality (≠), 7
  standard definition, 62
inorder, 118
input object, 6
input/output object, 6
InputType type function, 11
insertion in a sequence, 217
Integer concept, 18, 40
interpretation, 2
intersection (∩), 231
interval, 231
into transformation, 169
invariant, 148
  loop, 37
  recursion, 36
inverse of permutation, 170, 171
inverse_operation property, 66
isomorphic coordinate sets, 124
isomorphic types, 86
is_left_successor algorithm, 119
is_right_successor algorithm, 120
iterator adapter
  for bidirectional bifurcate coordinates,
    project, 124
  random access from indexed, 114
  reverse from bidirectional, 112
  underlying type, 224
Iterator concept, 91
iterator invalidation in array, 221
IteratorConcept type function, 187
IteratorType type function, 133, 134, 213
K
Kislitsyn, Sergei, 55
k_rotate_from_permutation_indexed
  algorithm, 180
k_rotate_from_permutation_random_access
  algorithm, 180
L
Lagrange, J.-L., 107
Lakshman, T. K., 159
largest_doubling algorithm, 75
less (<), 62
  for array_k, 212
  for bounded_range, 215
  less for TotallyOrdered, 130
natural total ordering, 61
  for pair, 210
less or equal (≤), 62
lexicographical_compare algorithm, 129
lexicographical_equal algorithm, 127
lexicographical Equivalent algorithm, 127
lexicographical_less algorithm, 130
limit in a range, 95
linear ordering, 52
Linearizable concept, 213
link rearrangement, 134
  on lists, 219
linked iterator, 133
linked structures, forward vs. bidirectional, 219

LinkedBifurcateCoordinate concept, 144
linker object, 133
linker_to_head machine, 139
linker_to_tail machine, 135
links, reversing, 145
list
  doubly linked, 218
  singly linked, 218
Lo, Raymond, 182
load, 4
local part of composite object, 217
local state, 6
locality of reference, 143
loop invariant, 37
lower bound, 107
lower_bound_n algorithm, 109
lower_bound_predicate algorithm, 108

M
machine, 120
  advance_tail, 135
  copy_backward_step, 154
  copy_step, 152
  count_down, 153
  linker_to_head, 139
  linker_to_tail, 135
merge_n_step.0 machine, 205
merge_n_step.1 machine, 205
merge_n_with_buffer algorithm, 202
mod (remainder), 18
model, partial, 70
models, 11
Module concept, 70
monoid, 67
multipass traversal, 106
MultiplicativeGroup concept, 68
MultiplicativeMonoid concept, 67
MultiplicativeSemigroup concept, 66
multiset, 227
Musser, David, 13
mutable range, 151
mutable_bounded_range property, 151
mutable_counted_range property, 151
mutable_weak_range property, 151
mutative rearrangement, 172

N
natural total ordering, < reserved for, 61
negative, 40
nil, 134
Noether, Emmy, 13
noncircularity of composite object, 216
none algorithm, 97
NonnegativeDiscreteArchimedeanSemiring concept, 86
nontotal procedure, 17
not (¬), 231
not_all algorithm, 97
not_overlapped property, 157
not_overlapped_backward property, 155
not_overlapped_forward property, 153
not_write_overlapped property, 159
null link, 218

O
object, 4
  area, 227
  equality, 5
  starting address, 216
state, 4
object type, 4
odd, 40
one, 40
one-to-one transformation, 169
onto transformation, 169
open interval, 231
Operation concept, 16
or (\(\lor\)), 231
orbit, 18–20
orbit_structure algorithm, 28
orbit_structure_nonterminating_orbit
algorithm, 27
OrderedAdditiveGroup concept, 70
OrderedAdditiveMonoid concept, 70
OrderedAdditiveSemigroup concept, 70
ordering, linear, 52
ordering-based rearrangement, 172
output object, 6
overloading, 43, 133, 144
own state, 6
ownership, of parts by composite
object, 216

P
pair type, 11, 209
parameter passing, 9
part of composite object, 215–219
partial model, 70
partial procedure, 17
partial (usage convention), 232
partially formed object state, 7
partially_associative property, 98
partition algorithm, origin of, 195
partition point, 105
  lower and upper bounds, 107
partition rearrangement, semistable, 192
partitioned property, 105
partitioned range, 105
partitioned_at_point algorithm, 191
partition_bidirectional algorithm, 194
partition_copy algorithm, 160
partition_copy_n algorithm, 160
partition_linked algorithm, 140
partition_point algorithm, 107
partition_point_n algorithm, 107
partition_semistable algorithm, 192
partition_single_cycle algorithm, 194
partition_stable_iterative algorithm, 201
partition_stable_n algorithm, 197
partition_stable_n_adaptive algorithm, 197
partition_stable_n_nonempty
algorithm, 197
partition_stable_singleton algorithm, 196
partition_stable_with_buffer algorithm, 195
partition_trivial algorithm, 198
permanently placed part of composite object, 217
permutation, 170
  composition, 170
cycle, 171
cyclic, 171
from, 172
index, 172
inverse, 170, 171
product of its cycles, 171
reverse, 174
rotation, 178
to, 172
transposition, 171
permutation group, 170
phased_applicator algorithm, 147
pivot, 205
position-based rearrangement, 172
positive, 40
postorder, 118
potential_partition_point algorithm, 191
power
  of associative operation \((a^n)\), 32
  powers of same element commute, 32
  of transformation \((f^n)\), 17
power algorithm, 42
  operation count, 34
power_accumulate algorithm, 41
power_accumulate_positive algorithm, 41
power_right_associated algorithm, 33
power_unary algorithm, 18
precedes preserving link rearrangement, 135
precedes or equal \((\preceq)\), 95
precondition, 13
predecessor
  of integer, 40
  of iterator, 111
Predicate concept, 15
predicate-based rearrangement, 172
predicate_source algorithm, 140
prefix of extent, 220
preorder, 118
prime property, 14
procedure, 6
  abstract, 13
  functional, 9
  nontotal, 17
  partial, 17
  total, 17
product (·)
  of integers, 18
  in multiplicative semigroup, 66
  in semimodule, 69
program transformation
  accumulation-variable elimination, 39
  accumulation-variable introduction, 35
  common-subexpression elimination, 35
  enabled by regular types, 35
  forward to backward iterators, 112
  relaxing precondition, 38
  strengthening precondition, 38
  strict tail-recursive, 37
  tail-recursive form, 35
project
  abstracting platform-specific copy
    algorithms, 164
  algorithms for bidirectional bifurcate
    algorithms, 123
  axioms for random-access iterator, 113
  benchmark and composite algorithm for
    rotate, 189
  concepts for bounded binary
    integers, 87
  coordinate structure concept, 131
  cross-type operations, 14
  cycle-detection algorithms, 29
  dynamic-sequences benchmark, 222
  dynamic-sequences implementation, 222
  dynamic-sequences interfaces, 222
  floating-point nonassociativity, 42
  isomorphism, equivalence, and ordering
    using tree_rotate, 148
  iterator adapter for bidirectional bifurcate
    coordinates, 124
  linear recurrence sequences, 47
  minimum-comparison stable sorting and
    merging, 61
  nonhalvable Archimedean monoids, 75
  order-selection stability, 61
  reallocation strategy for single-extent
    arrays, 221
  searching for a subsequence within a
    sequence, 114
  setting for Stein gcd, 81
  sorting library, 208
  underlying type used in major library, 225
  projection regularity, 216
  proper underlying type, 223
  properly partial object state, 5
  properly partial value type, 2
  property
    aliased, 150
    annihilation, 68
    associative, 31
    asymmetric, 50
    backward_offset, 161
    bounded_range, 93
    commutative, 66
    complement_of_converse, 104
    counted_range, 93
    discreteness, 85
    disjoint, 134
    distributive, 68
    equivalence, 51
    forward_offset, 162
    identity_element, 65
    identity_offset, 65
    increasing_counted_range, 105
    increasing_range, 105
    inverse_operation, 66
    mergeable, 203
    mutable_bounded_range, 151
    mutable_counted_range, 151
    mutable_weak_range, 151
    notation, 14
    not_overlapped, 157
    not_overlapped_backward, 155
    not_overlapped_forward, 153
    not_write_overlapped, 159
    partially_associative, 98
    partitioned, 105
    prime, 14
    readable_bounded_range, 95
    readable_counted_range, 96
    readable_tree, 123
    readable_weak_range, 96
    reflexive, 50
    regular_unary_function, 14
    relation_preserving, 103
property (cont.)
  strict, 50
  strictly_increasing_counted_range, 105
  strictly_increasing_range, 104
  symmetric, 50
  total_ordering, 51
  transitive, 49
  tree, 117
  trichotomy, 51
  weak trichotomy, 51
  weak_ordering, 52
  writable_range, 92
  writable_bounded_range, 150
  writable_counted_range, 150
  writable_weak_range, 150
  write_alias, 159
  proposition, independence of, 86
  pseudopredicate, 136
  pseudorelation, 137
  pseudotransformation, 91

Q
  quotient (/), of integers, 18
  quotient
    in Euclidean semimodule, 80
    in Euclidean semiring, 79
  QuotientType type function, 72
  quotient_remainder algorithm, 85
  quotient_remainder_nonnegative algorithm, 82
  quotient_remainder_nonnegative_iterative algorithm, 83

R
  random-access iterator, equivalent to indexed iterator, 113
  RandomAccessIterator concept, 113
  range
    backward movement, 112
    closed bounded ([f, l]), 94
    closed weak or counted ([f, n]), 94
    empty, 95
    half-open bounded ([f, l]), 94
    half-open weak or counted ([f, n]), 94
    increasing, 103
    limit, 95
    lower bound, 107
  mutable, 151
  partition point, 105
  partitioned, 105
  readable, 95
  size, 94
  strictly increasing, 103
  upper bound, 107
  writable, 150
  reachability
    of bifurcate coordinate, 116
    in orbit, 18
  reachable algorithm, 121
  readable range, 95
  readable_bounded_range property, 95
  readable_counted_range property, 96
  readable_tree property, 123
  readable_weak_range property, 96
  rearrangement, 172
    bin-based, 172
    copying, 172
    link, 134
    mutative, 172
    ordering-based, 172
    position-based, 172
    reverse, 174
    rotation, 179
    recursion invariant, 36
  reduce algorithm, 99
  reduce_balanced algorithm, 200
  reduce_nonempty algorithm, 99
  reduce_nonzeroes algorithm, 100
  reduction, 98
  reference counting, 230
  refinement of concept, 11
  reflexive property, 50
  Regular concept, 11
    and program transformation, 35
  regular function on value type, 3
  regular type, 6–8
  regularity, 216, 217
  regular Unary function property, 14
  Relation concept, 49
  relational concept, 69
  relationship, 229
  relation_preserving property, 103
  relation_source algorithm, 141
  relaxing precondition, 38
remainder
algorithm, 84
  in Euclidean semimodule, 80
  in Euclidean semiring, 79
remainder (mod), of integers, 18
remainder_nonnegative algorithm, 74
remainder_nonnegative_iterative algorithm, 75
remote part of composite object, 217
representation, 2
representational equality, 3, 228
requires clause, 13
  syntax, 240
resources, 4
result space, 10
returning useful information, 87, 96, 97,
  101–103, 106, 112, 152, 153, 159,
  163, 174, 179, 182, 211
reverse rearrangement, 174
reverse_append algorithm, 139, 140
reverse_bidirectional algorithm, 175
reverse_copy algorithm, 156
reverse_copy_backward algorithm, 156
reverse_copy_backward_step machine, 156
reverse_copy_step machine, 155
reverse_indexed algorithm, 186
reverse_n_adaptive algorithm, 178
reverse_n_bidirectional algorithm, 175
reverse_n_forward algorithm, 177
reverse_n_indexed algorithm, 175
reverse_n_with_buffer algorithm, 176
reverse_swap_ranges algorithm, 167
reverse_swap_ranges_bounded algorithm, 167
reverse_swap_ranges_n algorithm, 168
reverse_swap_step machine, 166
reverse_with_temporary_buffer algorithm,
  187, 225
reversing links, 145
Rhind Mathematical Papyrus
  division, 73
  power, 33
Ring concept, 69
rotate algorithm, 187
rotate_bidirectional_nontrivial algorithm, 182
rotate_cycles algorithm, 181
rotate_forward.annotated algorithm, 183
rotate_forward_nontrivial algorithm, 184
rotate_forward_step algorithm, 184
rotate_indexed_nontrivial algorithm, 181
rotate_nontrivial algorithm, 188
rotate_partial_nontrivial algorithm, 185
rotate.random_access_nontrivial algorithm,
  181
rotate_with_buffer_backward_nontrivial algorithm, 186
rotate_with_buffer_nontrivial algorithm, 185
rotation
  permutation, 178
  rearrangement, 179
S
  schema, concept, 124
Schreier, Jozef, 55
Schwarz, Jerry, 150
segmented array, 221
segmented index, 221
select_0.2 algorithm, 53, 63
select_0.3 algorithm, 54
select.1.2 algorithm, 54
select.1.3 algorithm, 55
select.1.3_ab algorithm, 55
select.1.4 algorithm, 56, 59
select.1.4_ab algorithm, 56, 59
select.1.4_ab_cd algorithm, 56, 58
select.2.3 algorithm, 54
select.2.5 algorithm, 60
select.2.5_ab algorithm, 60
select.2.5_ab_cd algorithm, 59
semi (usage convention), 232
semigroup, 66
Semimodule concept, 69
Semiring concept, 68
semistable partition rearrangement, 192
sentinel, 101
Sequence concept, 216
  extent-based models, 219
  linked models, 219
set, 231
single-ended array, 220
single-extent array, 220
single-extent index, 221
single-pass traversal, 91
singly linked list, 218
sink, 149
size
  for array_k, 212
  for bounded_range, 214
  for Linearizable, 213
size of an orbit, 20
size of a range, 94
SizeType type function, 213
slanted index, 221
slow_quotient algorithm, 73
slow_remainder algorithm, 72
snapshot, 1
some algorithm, 97
sort_linked_nonempty_n algorithm, 142
sort_n algorithm, 207
sort_n_adaptive algorithm, 207
sort_n_with_buffer algorithm, 203
source, 90
space complexity, memory adaptive, 177
species
  abstract, 2
  concrete, 2
splicing link rearrangement, 219
split_copy algorithm, 158
split_linked algorithm, 137
stability, 52
  of merge, 203
  of partition, 192
  of sort, 204
  of sort on linked range, 142
stability index, 53
Standard Template Library, x
starting address, 4, 216
state of object, 4
Stein, Josef, 81
Stein gcd, 81
STL, x
store, 4
strengthened relation, 53
strengthening precondition, 38
strict property, 50
strict tail-recursive, 37
strictly increasing range, 103
strictly_increasing_counted_range property, 105
strictly_increasing_range property, 104
structural equality, 228
subpart of composite object, 216
subset (⊂), 231
subtraction, in additive group, 67
subtractive_gcd algorithm, 78
subtractive_gcd_nonzero algorithm, 77
successor
  definition space on range, 94
  of integer, 40
  of iterator, 91
sum (+)
  in additive semigroup, 66
  of integers, 18
  of iterator and integer, 92
swap algorithm, 224
swap_basic algorithm, 223
swap_ranges algorithm, 165
swap_ranges_bounded algorithm, 166
swap_ranges_n algorithm, 166
swap_step machine, 165
symmetric complement of a relation, 52
symmetric property, 50
T
tail-recursive form, 35
technique. See program transformation
auxiliary computation during recursion, 176
memory-adaptive algorithm, 177
operation–accumulation procedure duality, 47
reduction to constrained subproblem, 54
returning useful information, 87, 96, 97, 101–103, 106, 112, 152, 153, 159, 163, 174, 179, 182, 211
transformation–action duality, 28
useful variations of an interface, 38
temporary_buffer type, 187
terminal element under transformation, 18
terminating algorithm, 23
three-valued compare, 63
Tighe, Joseph, 179
to-permutation, 172
total object state, 5
total procedure, 17
total value type, 2
TotallyOrdered concept, 62
total_ordering property, 51
trait class, 240
transformation, 17
  composing, 17, 32
cyclic element, 18
fixed point of, 170
identity, 170
into, 169
of program. See program transformation
one-to-one, 169
onto, 169
orbit, 18
power of \((f^n)\), 17
terminal element, 18
Transformation concept, 17
transitive property, 49
transpose_operation algorithm, 201
transposition, 171
traversal
multipass, 106
single-pass, 91
of tree, recursive, 119
traverse algorithm, 123
traverse_nonempty algorithm, 118
traverse_phased_rotating algorithm, 148
traverse_rotating algorithm, 146
traverse_step machine, 121
tree property, 117
tree_rotate machine, 145
trichotomy law, 51
trivial cycle, 171
twice, 40
two-pointer header doubly linked list, 218
type
array_k, 210
bounded_range, 214
computational basis, 6
counter_machine, 200
isomorphism, 86
models concept, 11
pair, 11, 209
regular, 6
temporary_buffer, 187
triple, 11
underlying_iterator, 225
visit, 118
type attribute, 10
Arity, 11
type concept, 11
type constructor, 11
type function, 11
Codomain, 11
DifferenceType, 113
DistanceType, 17, 91
Domain, 12
implemented via trait class, 240
InputType, 11
IteratorConcept, 187
IteratorType, 133, 134, 213
QuotientType, 72
SizeType, 213
UnderlyingType, 223
ValueType, 90, 149, 213
WeightType, 115
U
unambiguous value type, 3
UnaryFunction concept, 12
UnaryPredicate concept, 16
underlying type, 164, 223
iterator adapters, 224
proper, 223
UnderlyingType type function, 223
underlying_iterator type, 225
underlying_ref algorithm, 224
union \((\cup)\), 231
uniquely represented object type, 5
uniquely represented value type, 2
univalent concept, 86
upper bound, 107
upper_bound_n algorithm, 109
upper_bound_predicate algorithm, 109
useful variations of an interface, 38
usefulness of concept, 87
V
value, 2
value type, 2
ambiguous, 3
properly partial, 2
regular function on, 3
total, 2
uniquely represented, 2
ValueType type function, 90, 149, 213
visit type, 118
W
weak (usage convention), 232
weak-trichotomy law, 51
<table>
<thead>
<tr>
<th>Term</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>weakening of concept</td>
<td>11</td>
</tr>
<tr>
<td>weak_ordering property</td>
<td>52</td>
</tr>
<tr>
<td>weak_range property</td>
<td>92</td>
</tr>
<tr>
<td>weight algorithm</td>
<td>122</td>
</tr>
<tr>
<td>WeightType type function</td>
<td>115</td>
</tr>
<tr>
<td>weight_recursive algorithm</td>
<td>117</td>
</tr>
<tr>
<td>weight_rotating algorithm</td>
<td>147</td>
</tr>
<tr>
<td>well-formed object</td>
<td>5</td>
</tr>
<tr>
<td>well-formed value</td>
<td>2</td>
</tr>
<tr>
<td>words in memory</td>
<td>4</td>
</tr>
<tr>
<td>writable range</td>
<td>150</td>
</tr>
<tr>
<td>writable_bounded_range property</td>
<td>150</td>
</tr>
<tr>
<td>writable_counted_range property</td>
<td>150</td>
</tr>
<tr>
<td>writable_weak_range property</td>
<td>150</td>
</tr>
<tr>
<td>write_aliased property</td>
<td>159</td>
</tr>
<tr>
<td>Z</td>
<td></td>
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<tr>
<td>zero</td>
<td>40</td>
</tr>
</tbody>
</table>
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