

GEOMETRIC THINKING AND GEOMETRIC CONCEPTS

Chapter

7

Geometry in K–8 is finally being taken seriously. Geometry used to be the chapter that was skipped or put off until late in the year. Many teachers were not comfortable with geometry, associating it with high school and proofs. Nor was geometry seen as important because it was only minimally tested on standardized tests. Now geometry is a strand of the curriculum in nearly every state and district.

This change is due in large part to the influence of the NCTM standards movement beginning in 1989. A second significant influence is an attention to a theoretical perspective that has helped us understand how students reason about spatial concepts.

Geometry Goals for Your Students

It is useful to think about your geometry objectives in terms of two quite different yet related frameworks: spatial reasoning, or spatial sense, and the specific content such as that most likely found in your state or district objectives. The first of these frameworks has to do with the way students think and reason about shape and space. There is a well-researched theoretical basis for organizing the development of geometric thought that guides this framework. The second framework is content in the more traditional sense—knowing about

big ideas

- 1** What makes shapes alike and different can be determined by an array of geometric properties. For example, shapes have sides that are parallel, perpendicular, or neither; they have line symmetry, rotational symmetry, or neither; they are similar, congruent, or neither.
- 2** Shapes can be moved in a plane or in space. These changes can be described in terms of translations (slides), reflections (flips), and rotations (turns).
- 3** Shapes can be described in terms of their location in a plane or in space. Coordinate systems can be used to describe these locations precisely. In turn, the coordinate view of shape offers another way to understand certain properties of shapes, changes in position (transformations), and how they appear or change size (visualization).
- 4** Shapes can be seen from different perspectives. The ability to perceive shapes from different viewpoints helps us understand relationships between two- and three-dimensional figures and mentally change the position and size of shapes.

symmetry, triangles, parallel lines, and so forth. The NCTM *Principles and Standards for School Mathematics* authors have helped describe content goals across the grades. We need to understand both of these aspects of geometry—thought and content—so that we can best help students grow.

Spatial Sense

Spatial sense can be defined as an intuition about shapes and the relationships among shapes. Individuals with spatial sense have a feel for the geometric aspects of their surroundings and the shapes formed by objects in the environment.

Spatial sense includes the ability to mentally visualize objects and spatial relationships—to turn things around in your mind. It includes a comfort with geometric descriptions of objects and position. People with spatial sense appreciate geometric form in art, nature, and architecture. They are able to use geometric ideas to describe and analyze their world.

Many people say they aren't very good with shape or that they have poor spatial sense. The typical belief is that you are either born with spatial sense or not. This simply is not true! We now know that rich experiences with shape and spatial relationships, when provided consistently over time, can and do develop spatial sense. Without geometric experiences, most people do not grow in their spatial sense or spatial reasoning. Between 1990 and 1996, NAEP data indicated a steady, continuing improvement in students' geometric reasoning at all three grades tested, 4, 8, and 12 (Martin & Strutchens, 2000). Students did not just get smarter. More likely there has been an increasing emphasis on geometry at all grades. Still, much more needs to be done if U.S. children are to rise to the same level as their European and Asian counterparts.

Geometric Content

For too long, the geometry curriculum in the United States has been somewhat of an eclectic mix of activities and lists of “bold print words”—too much emphasis has been placed on learning terminology. At the same time, the growing emphasis placed on geometry has spawned a huge assortment of wonderful tasks for students. Fortunately, the authors of *Principles and Standards for School Mathematics* have provided a content framework for the pre-K–12 curriculum. As with each of the content standards, the geometry standard has a number of goals that apply to all grade levels. The four goals for geometry can be loosely summarized with these headings: *Shapes and Properties*, *Transformation*, *Location*, and *Visualization*. A very brief description of these headings is offered next.

- *Shapes and Properties* includes a study of the properties of shapes in both two and three dimensions, as well as a study of the relationships built on properties.
- *Transformation* includes a study of translations, reflections, and rotations (slides, flips, and turns) and the study of symmetries.
- *Location* refers primarily to coordinate geometry or other ways of specifying how objects are located in the plane or in space.
- *Visualization* includes the recognition of shapes in the environment, developing relationships between two- and three-dimensional objects, and the ability to draw and recognize objects from different perspectives.

The value of these content goals is that a content framework finally exists that cuts across grades so that both teachers and curriculum planners can examine growth from year to year.

You are strongly encouraged to read the geometry goals for grades pre-K–2 and 3–5 in *Principles and Standards* (NCTM, 2000).

Geometric Thought: Reasoning About Shapes and Relationships

Not all people think about geometric ideas in the same manner. Certainly, we are not all alike, but we are all capable of growing and developing in our ability to think and reason in geometric contexts. The research of two Dutch educators, Pierre van Hiele and Dina van Hiele-Geldof, has provided insight into the differences in geometric thinking and how the differences came to be. The van Hieles' work began in 1959 and immediately attracted a lot of attention in the Soviet Union but for nearly two decades received little notice in this country (Hoffer, 1983; Hoffer & Hoffer, 1992). But today the van Hiele theory has become the most influential factor in the American geometry curriculum.

The van Hiele Levels of Geometric Thought

The most prominent feature of the model is a five-level hierarchy of ways of understanding spatial ideas. Each of the five levels describes the thinking processes used in geometric contexts. The levels describe how we think, and what types of geometric ideas we think about, rather than how much knowledge we have. A significant difference from one level to the next is the objects of thought—what we are able to think about geometrically.

Level 0: Visualization

The objects of thought at level 0 are shapes and what they “look like.”

Students recognize and name figures based on the global, visual characteristics of the figure—a gestaltlike approach to shape. Students operating at this level are able to make measurements and even talk about properties of shapes, but these properties are not abstracted from the shapes at hand. It is the appearance of the shape that defines it for the student. A square is a square “because it looks like a square.” Because appearance is dominant at this level, appearances can overpower properties of a shape. For example, a square that has been rotated so that all sides are at a 45-degree angle to the vertical may now be a diamond and no longer a square. Students at this level will sort and classify shapes based on their appearances—“I put these together because they are all pointy” (or “fat,” or “look like a house,” or are “dented in sort of,” and so on). With a focus on the appearances of shapes, students are able to see how shapes are alike and different. As a result, students at this level can create and begin to understand classifications of shapes.

The products of thought at level 0 are classes or groupings of shapes that seem to be “alike.”

Level 1: Analysis

The objects of thought at level 1 are classes of shapes rather than individual shapes.

Students at the analysis level are able to consider all shapes within a class rather than a single shape. Instead of talking about *this* rectangle, it is possible to talk about *all* rectangles. By focusing on a class of shapes, students are able to think about what makes a rectangle a rectangle (four sides, opposite sides parallel, opposite sides same length, four right angles, congruent diagonals, etc.). The irrelevant features (e.g., size or orientation) fade into the background. At this level, students begin to appreciate that a collection of shapes goes together because of properties. Ideas about an individual shape can now be generalized to all shapes that fit that class. If a shape belongs to a particular class such as cubes, it has the corresponding properties of that class. “All cubes have six congruent faces, and each of those faces is a square.” These properties were only implicit at level 0. Students operating at level 1 may be able to list all the properties of squares, rectangles, and parallelograms but not see that these are subclasses of one another, that all squares are rectangles and all rectangles are parallelograms. In defining a shape, level 1 thinkers are likely to list as many properties of a shape as they know.

The products of thought at level 1 are the properties of shapes.

Level 2: Informal Deduction

The objects of thought at level 2 are the properties of shapes.

As students begin to be able to think about properties of geometric objects without the constraints of a particular object, they are able to develop relationships between and among these properties. “If all four angles are right angles, the shape must be a rectangle. If it is a square, all angles are right angles. If it is a square, it must be a rectangle.” It is at this level that students can appreciate the nature of a definition. With greater ability to engage in “if-then” reasoning, shapes can be classified using only minimum characteristics. For example, four congruent sides and at least one right angle can be sufficient to define a square. Rectangles are parallelograms with a right angle. Observations go beyond properties themselves and begin to focus on logical arguments *about* the properties. Students at level 2 will be able to follow and appreciate an informal deductive argument about shapes and their properties. Proofs may be more intuitive than rigorously deductive. However, there is an appreciation that a logical argument is compelling. An appreciation of the axiomatic structure of a formal deductive system, however, remains under the surface.

The products of thought at level 2 are relationships among properties of geometric objects.

Level 3: Deduction

The objects of thought at level 3 are relationships among properties of geometric objects.

At level 3, students are able to examine more than just the properties of shapes. Their earlier thinking has produced conjectures concerning relationships among properties. Are these conjectures correct? Are they “true”? As this analysis of the informal arguments takes place, the structure of a system complete with axioms, definitions, the-

orems, corollaries, and postulates begins to develop and can be appreciated as the necessary means of establishing geometric truth. The student at this level is able to work with abstract statements about geometric properties and make conclusions based more on logic than intuition. This is the level of the traditional high school geometry course.

The products of thought at level 3 are deductive axiomatic systems for geometry.

Level 4: Rigor

The objects of thought at level 4 are deductive axiomatic systems for geometry.

At the highest level of the van Hiele hierarchy, the objects of attention are axiomatic systems themselves, not just the deductions within a system. This is generally the level of a college mathematics major who is studying geometry as a branch of mathematical science.

The products of thought at level 4 are comparisons and contrasts among different axiomatic systems of geometry.

We have given brief descriptions of all five levels to illustrate the scope of the van Hiele theory. In every grade from 5 to 8, you will certainly see students at levels 0, 1, and 2.

Characteristics of the van Hiele Levels

You no doubt noticed that the products of thought at each level are the same as the objects of thought at the next. This object–product relationship between levels of the van Hiele theory is illustrated in Figure 7.1. The objects (ideas) must be created at one level so that relationships among these objects can become the focus of the next level. In addition to this key concept of the theory, four related characteristics of the levels of thought merit special attention.

1. The levels are sequential. To arrive at any level above level 0, students must move through all prior levels. To move through a level means that one has experienced geometric thinking appropriate for that level and has created in one's own mind the types of objects or relationships that are the focus of thought at the next level.

The van Hiele Theory of Geometric Thought

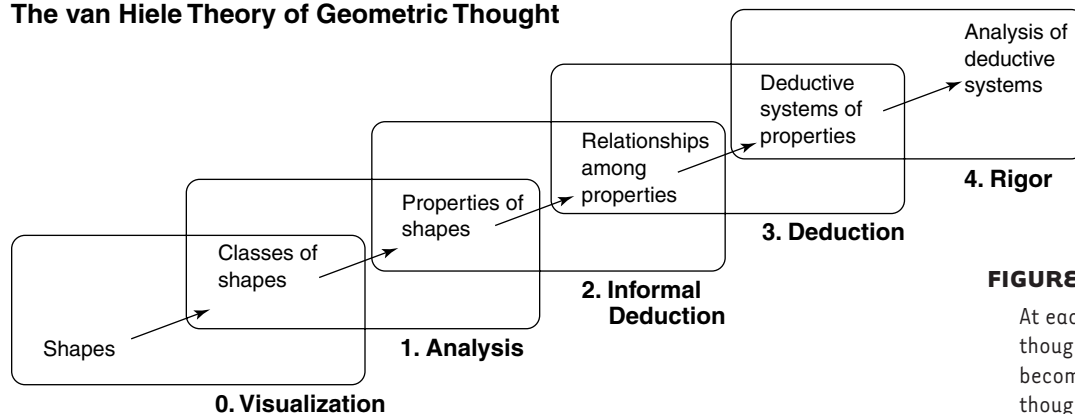


FIGURE 7.1

At each level of geometric thought, the ideas created become the focus or object of thought at the next level.

2. The levels are not age dependent in the sense of the developmental stages of Piaget. A third grader or a high school student could be at level 0. Indeed, some students and adults remain forever at level 0, and a significant number of adults never reach level 2. But age is certainly related to the amount and types of geometric experiences that we have. Therefore, it is reasonable to assume that most children in the K–2 range as well as many children in grades 3 and 4 are at level 0.
3. Geometric experience is the greatest single factor influencing advancement through the levels. Activities that permit children to explore, talk about, and interact with content at the next level, while increasing their experiences at their current level, have the best chance of advancing the level of thought for those children. Some researchers believe that it is possible to be at one level with respect to a familiar area of content and at a lower level with less familiar ideas (Clements & Battista, 1992).
4. When instruction or language is at a level higher than that of the student, there will be a lack of communication. Students required to wrestle with objects of thought that have not been constructed at the earlier level may be forced into rote learning and achieve only temporary and superficial success. A student can, for example, memorize that all squares are rectangles without having constructed that relationship. A student may memorize a geometric proof but fail to create the steps or understand the rationale involved (Fuys, Geddes, & Tischler, 1988; Geddes & Fortunato, 1993).

Implications for Instruction

If the van Hiele theory is correct—and there is much evidence to support it—then a major goal of the K–8 curriculum must be to advance students’ level of geometric thought. If students are to be adequately prepared for the deductive geometry curriculum of high school, then it is important for their thinking to have grown to level 2 by the end of the eighth grade.

Not every teacher will be able to move children to the next level. However, all teachers should be aware that the experiences they provide are the single most important factor in moving children up this developmental ladder. Every teacher should be able to see some growth in geometric thinking over the course of the year.

The van Hiele theory and the developmental perspective of this book highlight the necessity of teaching at the child’s level of thought. However, almost any activity can be modified to span two levels of thinking, even within the same classroom. For many activities, how we interact with individual children will adapt the activity to their levels and encourage them or challenge them to operate at the next higher level.

Explorations help develop relationships. The more students play around with the ideas in activities, the more relationships they will discover. However, students need to learn how to explore ideas in geometry and play around with the relationships in order for ideas to develop and become meaningful.

The following sections contain descriptions of the types of activity and questioning that are appropriate for the first three levels. Apply these descriptors to the tasks that you pose to students and use them to guide your interaction with students. The use of physical materials, drawings, and computer models is a must at every level.

Instruction at Level 0

Instructional activities in geometry appropriate for level 0 should:

- Involve lots of sorting and classifying. Seeing how shapes are alike and different is the primary focus of level 0. As students learn more content, the types of things that they notice will become more sophisticated. At an early stage they may talk about very nongeometric-sounding attributes of shape such as “fat” or even the color of the pieces. When properties such as symmetry and numbers of sides and corners are introduced, students should be challenged to use these features to classify shapes.
- Include a sufficient variety of examples of shapes so that irrelevant features do not become important. Students need ample opportunities to draw, build, make, put together, and take apart shapes in both two and three dimensions. These activities should be built around specific characteristics or properties so that students develop an understanding of geometric properties and begin to use them naturally.

To help students move from level 0 to level 1, students should be challenged to test ideas about shapes for a variety of examples from a particular category. Say to them, “Let’s see if that is true for other rectangles,” or “Can you draw a triangle that does *not* have a right angle?” In general, students should be challenged to see if observations made about a particular shape apply to other shapes of a similar kind.

Instruction at Level 1

Instructional activities in geometry appropriate for level 1 should:

- Focus more on the properties of figures rather than on simple identification. As new geometric concepts are learned, the number of properties that figures have can be expanded.
- Apply ideas to entire classes of figures (e.g., *all* rectangles, *all* prisms) rather than individual models. Analyze classes of figures to determine new properties. For example, find ways to sort all possible triangles into groups. From these groups, define types of triangles. Dynamic geometry software such as *The Geometer’s Sketchpad* (Key Curriculum Press) is especially useful for exploring many examples of a class of shapes. This software is essential for exploring geometric ideas in grades 5 to 8.

To assist students in moving from level 1 to level 2, challenge them with questions such as “Why?” and those that involve some reasoning. For example, ask “If the sides of a four-sided shape are all congruent, will you always have a square?” and “Can you find a counterexample?”

Instruction at Level 2

Instructional activities in geometry appropriate for level 2 should:

- Encourage the making and testing of hypotheses or conjectures. “Do you think that will work all the time?” “Is that true for all triangles or just equilateral ones?”

- Examine properties of shapes to determine necessary and sufficient conditions for different shapes or concepts. “What properties of diagonals do you think will guarantee that you will have a square?”
- Use the language of informal deduction: *all, some, none, if-then, what if*, and so on.
- Encourage students to attempt informal proofs. As an alternative, require them to make sense of informal proofs that other students or you have suggested.

Assessment Note



Nearly all students in grades K–3 will be at level 0. However, by at least grade 3 teachers certainly want to begin to challenge students who seem able to engage in level 1 thinking. Teachers in the upper grades may have children at two or even all three levels within the same classroom. How do you discover the level of each student? Once you know, how will you select the right activities to match your students’ levels?

No simple test exists to pigeonhole students at a certain level. However, examine the descriptors for the first two levels. As you conduct an activity, listen to the types of observations that students make. Can they talk about shapes as classes? Do they refer, for example, to “rectangles” rather than basing discussion around a particular rectangle? Do they generalize that certain properties are attributable to a type of shape or simply the shape at hand? Do they understand that shapes do not change when the orientation changes? With simple observations such as these, you will soon be able to distinguish between levels 0 and 1.

At the upper grades, attempt to push students from level 1 to level 2. If students are not able to follow or appreciate logical arguments and are not comfortable with conjectures and if-then reasoning, these students are likely still at level 1 or below.

Content and the Levels of Thinking

This chapter offers a sample of activities organized around the four content areas: Shapes and Properties, Transformations, Location, and Visualization. A section of the chapter is devoted to each of these areas. The van Hiele theory applies to all geometric activity regardless of content. However, it is within the content area of shapes and properties that the theory is most clearly seen. For that reason the activities in that section are subdivided into those appropriate for level 0, level 1, and level 2 thinkers. You will find this subdivision helpful for matching activities to your students and encouraging student development of thinking to higher levels. The three remaining sections focus on activities for developing spatial sense through location, transformation, and visualization. Each of these sections is organized in a progression of difficulty and sophistication.

Understand that all of these subdivisions are quite fluid; that is, the content areas overlap and build on each other. Activities in one section may help develop geometric thinking in another area. For example, developing spatial sense through an investigation of symmetry can help students move from level 0 to level 1. A more sophisticated

analysis of symmetry can continue to help students move to level 2. In most instances, an activity described for one level of thinking can easily be adapted to an adjacent level simply by the way it is presented to students.

Shapes and Properties Activities

- Children need experiences with a rich variety of both two- and three-dimensional shapes. It is useful for students to be able to identify common shapes, notice likenesses and differences among shapes, become aware of the properties that different shapes have, and eventually use these properties to further define and understand their geometric world. As students find out more about shapes over time, they can begin to appreciate how definitions of special shapes come to be.

This gradual development of student understanding of shapes and their properties clearly reflects the van Hiele theory of geometric thought. Within this aspect of geometric content, an awareness and application of the theory to your instruction are most important.

Activities for Level 0 Thinkers

The emphasis at level 0 is on the shapes that students can observe, feel, build, take apart, and perceive in many ways. The general goal is to explore how shapes are alike and different and use these ideas to create classes of shapes (both physically and mentally). Some of these classes of shapes have names—rectangles, triangles, prisms, cylinders, and so on. Properties of shapes, such as parallel sides, symmetry, right angles, and so on, are included at this level but only in an informal, observational manner.

Children need experience with a rich variety of both two- and three-dimensional shapes. Triangles should be more than just equilateral. Shapes should have curved sides, straight sides, and combinations of these. Along the way, the names of shapes and their properties can be introduced casually but only after students have described the shape or property.

Remember that *level 0* is not a synonym for *primary*. If you teach in the upper grades, you will almost certainly have students who need to begin with activities similar to these.

Sorting and Classifying

As students work at classification of shapes, be prepared for them to notice features that you do not consider to be “real” geometric attributes, such as “curvy” or “looks like a rocket.” Students at this level will also attribute to shapes ideas that are not part of the shape, such as “points up” or “has a side that is the same as the edge of the board.”

For variety in two-dimensional shapes, create your own materials. A good set found in the Blackline Masters is called 2-D Shapes. Make multiple copies so that groups of students can all work with the same shapes. The shapes in Figure 7.2 are similar to those in the Blackline Masters, but you will want many more. Once you have your sets constructed, the following activities provide several ideas.

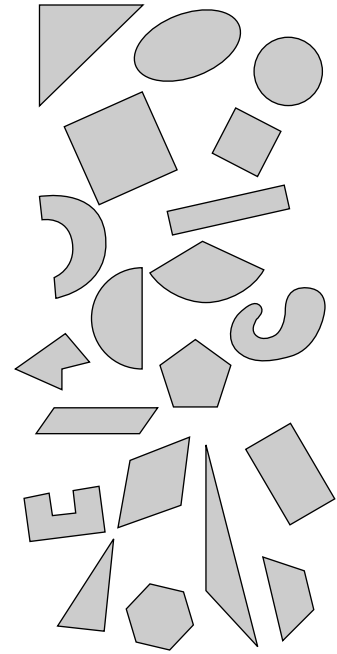


FIGURE 7.2
An assortment of shapes for sorting.



BLMs 12–18

ACTIVITY 7.1**Shape Sorts**

Have students work in groups of four with a set of 2-D Shapes similar to those in Figure 7.2. Here are several related activities that might be done in order:

- Each student randomly selects a shape. In turn, students tell one or two things they find interesting about their shape. There are no right or wrong responses.
- Students each randomly select two shapes. The task is to find something that is alike about their two shapes and something that is different. (Have them select their shapes before they know the task.)
- The group selects one shape at random and places it in the center of the workspace. Their task is to find all other shapes that are like the target shape, but all according to the same rule. For example, if they say, "This one is like our shape because it has a curved side and a straight side," then all other shapes that they put in the collection must have these properties. Challenge them to do a second sort with the same target shape but using a different property.
- Have students share their sorting rules with the class and show examples. All students then draw a new shape that will also fit in the group according to the same rule. They should write about their new shape and why it fits the rule.
- Do a "secret sort." You or one of the students creates a small collection of about five shapes that fit a secret rule. Leave others that belong in your group in the pile. The other students try to find additional pieces that belong to the set and/or guess the secret rule.



Why do you think that the teacher should not say things such as, "Find all the pieces with straight sides," or "Find the triangles," and instead let students choose how to sort?

In any sorting activity, the students should decide how to sort, not the teacher. This allows the students to do the activity using ideas *they* own and understand. By listening to the kinds of attributes that they use in their sorting, you will be able to tell what properties they know and use and how they think about shapes. At level 0, students will likely make groups such as "has a curved side" or will sort by the number of sides a shape has. They will use expressions such as "has sides that go the same way" for parallel, "dented in" for concave, and "square corners" for right angles.

The secret sorting activity is one option for introducing a new property. For example, sort the shapes so that all have at least one right angle or "square corner." When students discover your rule, you have an opportunity to talk more about that property.

The following activity is also done with the 2-D Assorted Shapes.

ACTIVITY 7.2**What's My Shape?**

From the Blackline Masters, make a set of 2-D Shapes on paper. Cut out about a third of the shapes and paste each inside a folded half-sheet of construction paper to make "secret shape" folders.

In a group, one student is designated the leader and given a secret-shape folder. The other students are to find the shape that matches the shape in the folder. To this end, they ask questions to which the leader can answer only “yes” or “no.” The students can sort the shapes as they ask questions to help narrow down the possibilities. They are not allowed to point to a piece and ask, “Is it this one?” Rather, they must continue to ask questions that reduce the choices to one shape. The final piece is tested against the one in the leader’s folder.

The difficulty of Activity 7.2 largely depends on the shape in the folder. The more shapes in the collection that resemble the secret shape, the more difficult the task.

Most of the activities in “Shape Sorts” can and should be done with three-dimensional shapes as well. The difficulty is finding or making a collection that has sufficient variability. Geoblocks are a large set of wooden blocks available through various distributors. The variety is good, but no blocks have curved surfaces. Check catalogs for other collections. Consider combining several different sets to get variation. Another option is to collect real objects such as cans, boxes, balls, and Styrofoam shapes. Again, the types of categories that students use will be quite revealing.



Assessment Note

The way that children describe shapes in “Shape Sorts” and similar activities with three-dimensional shapes is a good clue to their level of thinking. The classifications made by level 0 thinkers will generally be restricted to the shapes that they can actually put into a group. As they begin to think in terms of the properties of shapes, they will create categories based on properties, and their language will indicate that there are many more shapes in the group than those that are physically present. For example, students may say, “These shapes have square corners sort of like rectangles,” or “These look like boxes. All the boxes have square (rectangular) sides.”

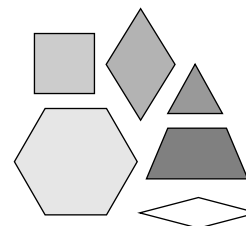
Constructing and Dissecting Shapes

Students need to freely explore how shapes fit together to form larger shapes and how larger shapes can be made of smaller shapes. Among two-dimensional shapes for these activities, pattern blocks and tangrams are the best known. In a 1999 article, Pierre van Hiele describes an interesting set of tiles he calls the mosaic puzzle (see Figure 7.3). Another excellent tile set for building is a set of triangles cut from squares (isosceles right triangles). Patterns for the mosaic puzzle and tangrams can be found in the Blackline Masters.

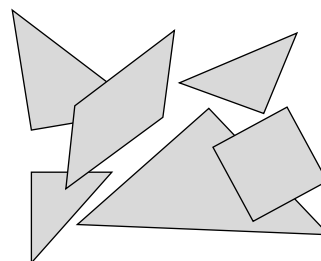
Although tangrams are extremely popular, their value begins to diminish in the intermediate grades. (An



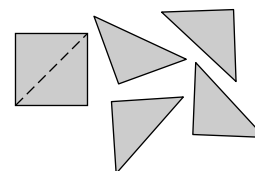
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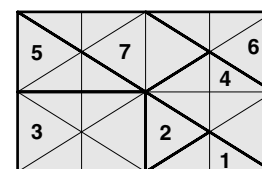
Pattern blocks



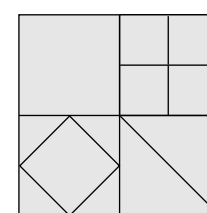
Tangrams



Triangles cut from squares



The 7-piece mosaic puzzle is built on an isometric grid (van Hiele, 1999).



Try cutting up squares or rectangles in other ways to get pieces that are related (Lindquist, 1987b).

FIGURE 7.3

Activities with tiles can involve an assortment of shapes or can be designed with just one shape.

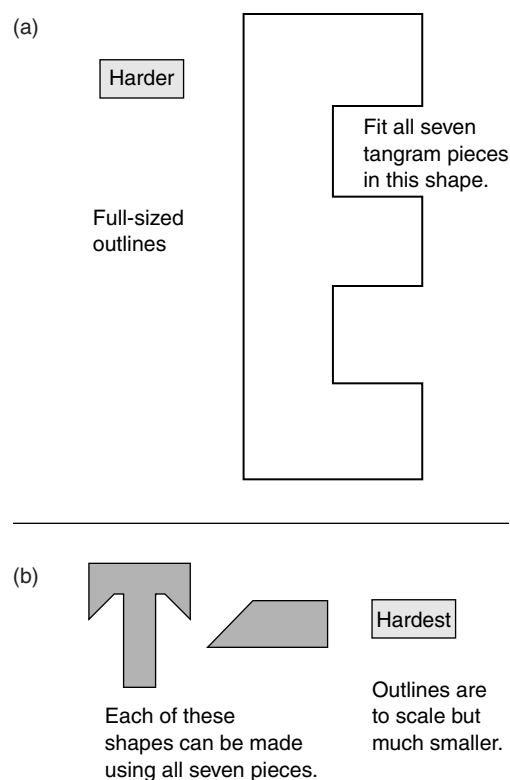


FIGURE 7.4

Two types of tangram puzzles that are different in difficulty.

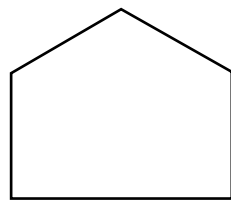
exception is their use in measuring area. See Chapter 8.) Nonetheless, level 0 students do gain experiences with the way that shapes fit together when they solve tangram puzzles. A full-sized outline that will contain exactly all seven tangram pieces can be quite challenging. The most difficult tangram puzzle is a shape that can be made of all seven pieces but shown to the student in reduced form. (See Figure 7.4.) This latter puzzle format involves proportional reasoning because the student must mentally enlarge the shape in order to create it with the tangrams.

The value of van Hiele's mosaic puzzle is partly due to the fact that the set contains five different angles (see Figure 7.5). If appropriate, you can use the pieces to talk about square corners (*right* angles) and angles that are more and less than a right angle (*obtuse* and *acute* angles).

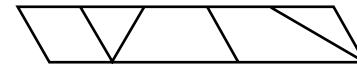
Building three-dimensional shapes is a little more difficult compared with two-dimensional shapes. A variety of commercial materials permits fairly creative construction of geometric solids (for example, 3D Geoshapes, Polydron, and the Zome System). The 3D Geoshapes and Polydron are examples of materials consisting of plastic polygons that snap together to make three-dimensional models. The Zome System is a stick and connector set; skeletal models can be created with a great deal of variation. The following are three highly recommended homemade approaches to skeletal models.

- *Plastic coffee stirrers with pipe cleaners.* Plastic stirrers can be easily cut to different lengths. To connect the corners, cut pipe cleaners into 2-inch lengths. These are inserted into the ends of the stirrers.
- *Plastic drinking straws with flexible joints.* Cut the straws lengthwise with scissors from the top down to the flexible joint. These slit ends can then be inserted into the uncut bottom ends of other straws, making a strong but flexible joint. Three or more straws are joined in this fashion to form two-dimensional polygons. To make skeletal solids, use tape or wire twist ties to join polygons side to side.
- *Rolled newspaper rods.* Fantastic superlarge skeletons can be built using newspaper and masking tape. Roll three large sheets of newspaper on the diagonal to form a rod. The more tightly the paper is rolled, the less likely the rod is to bend. Secure the roll at the center with a bit of masking tape. The ends of the rods are thin and flexible for about 6 inches where there is less paper. Connect the rods by bunching this thin part together and fastening with tape. Use masking tape freely, wrapping it several times around each joint. Additional rods can be joined after two or three are already taped (see Figure 7.6).

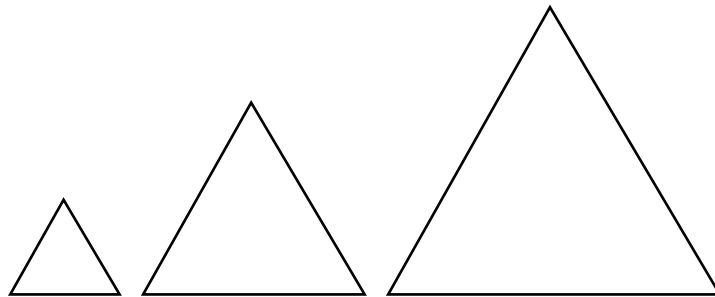
With these homemade models, students should compare the rigidity of a triangle with the lack of rigidity of polygons with more than three sides. Point out that triangles are used in many bridges, in the long booms of construction cranes, in gates, and in the structural parts of buildings. Discuss why this may be so. As students build large skeleton structures, they will find that they need to add diagonal members to form triangles. The more triangles, the less likely their structure will collapse.



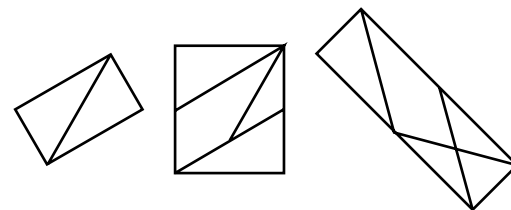
Make this house with two pieces. Now do it a different way. Can you make it with three pieces? How many ways? What about four pieces?



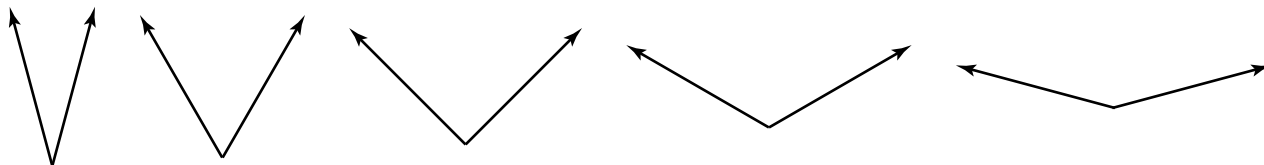
This is a long parallelogram. What other parallelograms can you make?



Build enlargements of the equilateral triangle.



How many different rectangles can you make? Can you make any in more than one way?

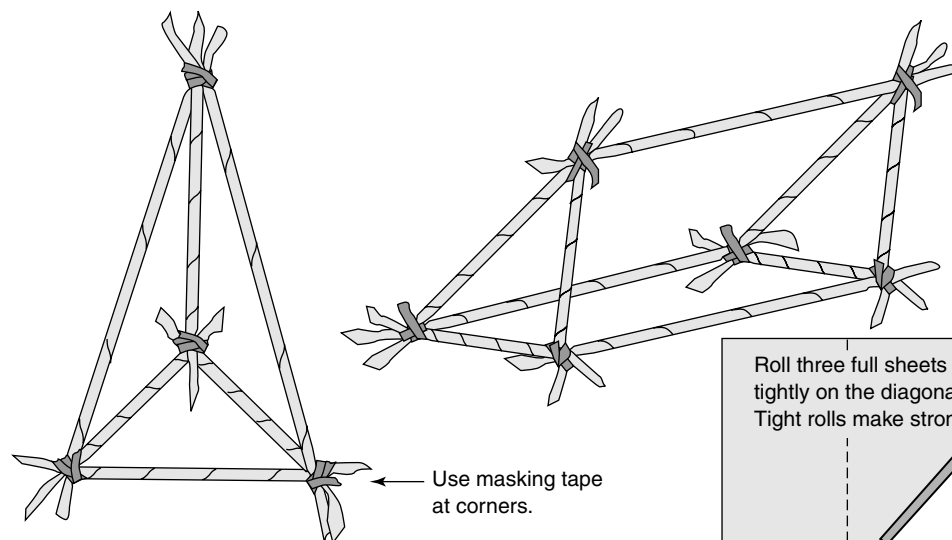


How many different-sized angles can you find in the set of pieces? Put them in order from smallest to largest.

FIGURE 7.5

A sample of activities with the mosaic puzzle.

Based on van Hiele, P. M. (1999). Developing geometric thinking through activities that begin with play. *Teaching Children Mathematics*, 5, 310–316.



Use masking tape at corners.

Roll three full sheets of newspaper very tightly on the diagonal. Secure with tape. Tight rolls make stronger sticks.

FIGURE 7.6

Large skeletal structures and special shapes can be built with tightly rolled newspaper. Young children can build free-form sculptures. Older children can be challenged to build shapes with specific properties. Overlap the ends about 6 inches to ensure strength.

The newspaper rod method is exciting because the structures quickly become large. Let students work in groups of four or five. They will soon discover what makes a structure rigid and ideas of balance and form. Students can be challenged to make shapes with specific properties. (See pp. 226–227.)

Tessellations

A *tessellation* is a tiling of the plane using one or more shapes in a repeated pattern with no holes or gaps. Making tessellations is an artistic way for level 0 and level 1 students from first grade to eighth grades to explore patterns in shapes and to see how shapes combine to form other shapes. One-shape or two-shape tessellation activities can vary considerably in difficulty. Some shapes are easier to tessellate than others (see Figure 7.7). When the shapes can be put together in more than one pattern, both the problem-solving level and the creativity increase. Literally hundreds of shapes can be used as tiles for tessellations.

Most students will benefit from using actual tiles to create patterns. Simple construction paper tiles can be cut quickly on a paper cutter. Other tiles can be traced onto construction paper and several thicknesses cut at once with scissors. When the tile shape fits in a grid, students can use dot or line grids and plan their tessellations with pencil and paper. To plan a tessellation, use only one color so that the focus is on the spatial relationships. To complete an artistic-looking tessellation, add a color design. Color designs are also repeated regularly all over the tessellation.

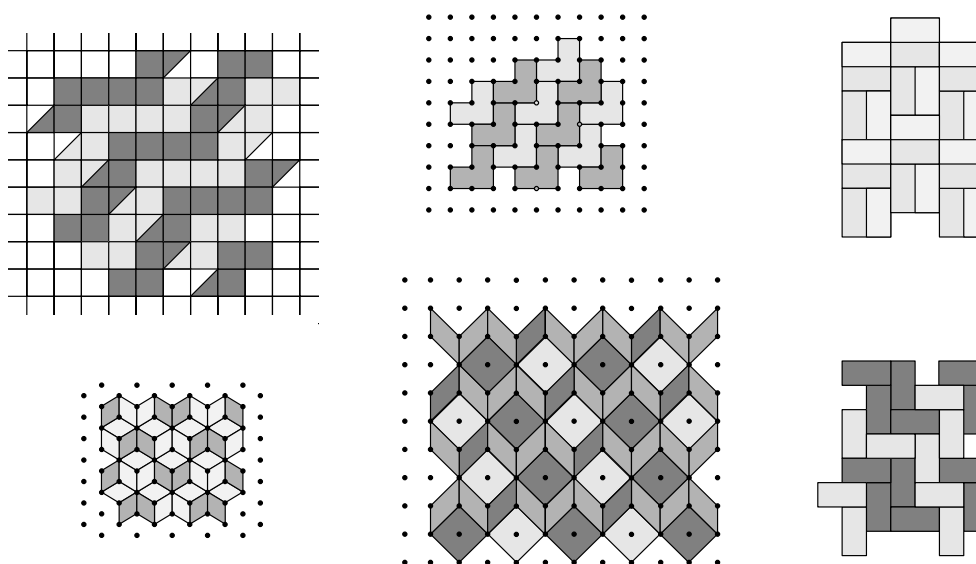
Tessellations can be made by gluing paper tiles to large sheets of paper, by drawing them on dot or line grids, or by tracing around a poster board tile. Work from the center out, leaving ragged edges to indicate that the pattern goes on and on.



Look at the top-left tessellation in Figure 7.7. What single tile (a combination of squares and half squares) made this pattern?

FIGURE 7.7

Tessellations.



Tessellations can be drawn on grids or made of construction paper tiles. They are challenging and provide an opportunity for both artistic creativity and spatial reasoning.

Activities for Level 1 Thinkers

A significant difference between level 1 and level 0 is the object of students' thought. While students will continue to use models and drawings of shapes, they begin to see these as representatives of classes of shapes. Their understanding of the properties of shapes—such as symmetry, perpendicular and parallel lines, and so on—continues to be refined.

For the sake of clarity, the important definitions of two- and three-dimensional shapes are provided here. You will notice that shape definitions include relationships between and among shapes.

Special Categories of Two-Dimensional Shapes

Table 7.1 lists some important categories of two-dimensional shapes. Examples of these shapes can be found in Figure 7.8.

In the classification of quadrilaterals and parallelograms, the subsets are not all disjoint. For example, a square is a rectangle and a rhombus. All parallelograms are trapezoids, but not all trapezoids are parallelograms.* Children at level 1 have difficulty seeing this type of subrelationship. They may quite correctly list all the properties of a square, a rhombus, and a rectangle and still identify a square as a “nonrhombus” or a “nonrectangle.” Is it wrong for students to refer to subgroups as disjoint sets? By fourth or fifth grade, it is only wrong to encourage such thinking. Burger (1985) points out that upper elementary students correctly use such classification schemes in other contexts. For example, individual students in a class can belong to more than one club. A square is an example of a quadrilateral that belongs to two other clubs.

Special Categories of Three-Dimensional Shapes

Important and interesting shapes and relationships also exist in three dimensions. Table 7.2 describes classifications of solids. Figure 7.9 on p. 196 shows examples of cylinders and prisms.

*Some definitions of trapezoid specify *only one* pair of parallel sides, in which case parallelograms would not be trapezoids. The University of Chicago School Mathematics Project (UCSMP) uses the “at least one pair” definition, meaning that parallelograms and rectangles are trapezoids.

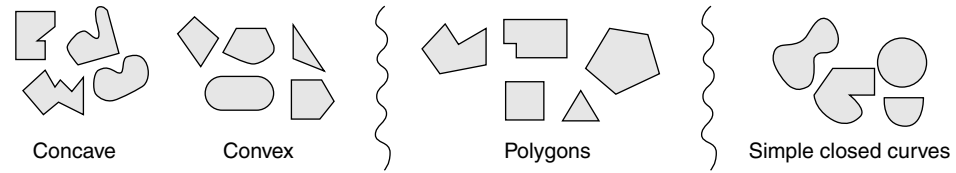
TABLE 7.1 Categories of Two-Dimensional Shapes

Shape	Description
Simple Closed Curves	
Concave, convex	An intuitive definition of <i>concave</i> might be “having a dent in it.” If a simple closed curve is not concave, it is <i>convex</i> . A more precise definition of <i>concave</i> may be interesting to explore with older students.
Symmetrical, nonsymmetrical	Shapes may have one or more lines of symmetry and may or may not have rotational symmetry. These concepts will require more detailed investigation.
Polygons	Simple closed curves with all straight sides.
Concave, convex	
Symmetrical, nonsymmetrical	
Regular	All sides and all angles are congruent.
Triangles	
Triangles	Polygons with exactly three sides.
<i>Classified by sides</i>	
Equilateral	All sides are congruent.
Isosceles	At least two sides are congruent.
Scalene	No two sides are congruent.
<i>Classified by angles</i>	
Right	Has a right angle.
Acute	All angles are smaller than a right angle.
Obtuse	One angle is larger than a right angle.
Convex Quadrilaterals	
Convex quadrilaterals	Convex polygons with exactly four sides.
Kite	Two opposing pairs of congruent adjacent sides.
Trapezoid	At least one pair of parallel sides.
Isosceles trapezoid	A pair of opposite sides is congruent.
Parallelogram	Two pairs of parallel sides.
Rectangle	Parallelogram with a right angle.
Rhombus	Parallelogram with all sides congruent.
Square	Parallelogram with a right angle and all sides congruent.

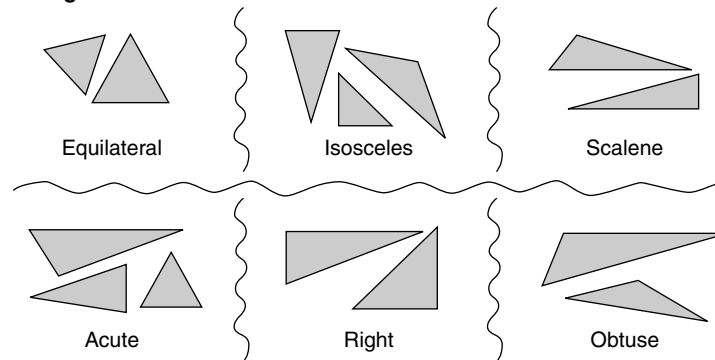
FIGURE 7.8

Classification of two-dimensional shapes.

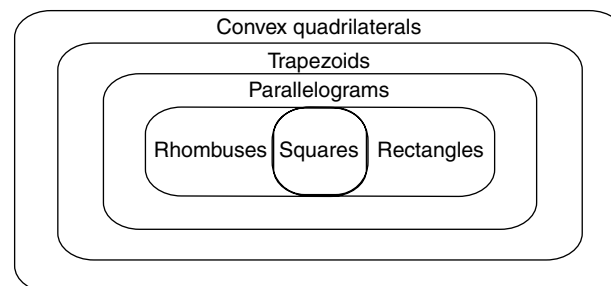
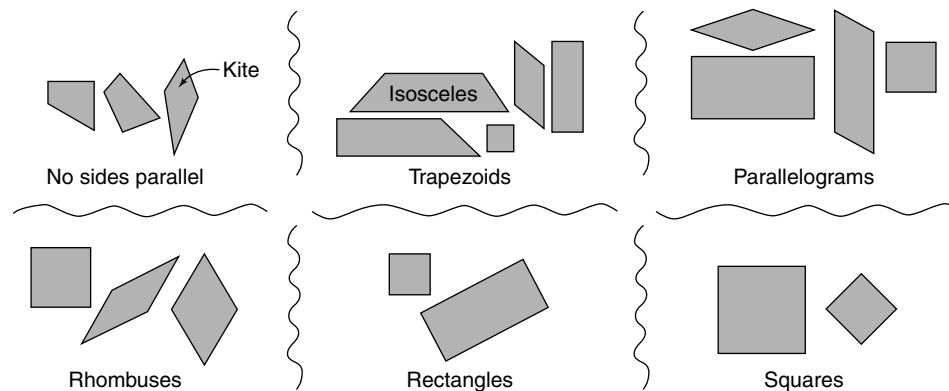
Simple Closed Curves



Triangles



Convex Quadrilaterals



There are many ways to sort polygons. Many with three and four sides have special names.

Level 1 thinking does not recognize these subrelationships.

TABLE 7.2
Categories of Three-Dimensional Shapes

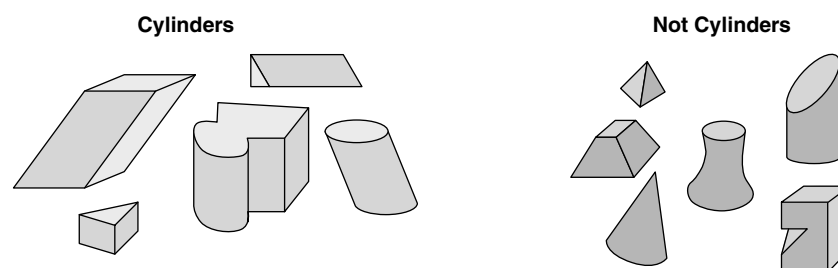
Shape	Description
Sorted by Edges and Vertices	
Sphere and “egglike” shapes	Shapes with no <i>edges</i> and no <i>vertices</i> (corners). Shapes with <i>edges</i> but no <i>vertices</i> (e.g., a flying saucer). Shapes with <i>vertices</i> but no <i>edges</i> (e.g., a football).
Sorted by Faces and Surfaces	
Polyhedron	Shapes made of all faces (a <i>face</i> is a flat surface of a solid). If all surfaces are faces, all the edges will be straight lines. Some combination of faces and rounded surfaces (cylinders are examples, but this is not a definition of a cylinder). Shapes with curved surfaces. Shapes with and without edges and with and without vertices. Faces can be parallel. Parallel faces lie in places that never intersect.
Cylinders	
Cylinder	Two congruent, parallel faces called <i>bases</i> . Lines joining corresponding points on the two bases are always parallel. These parallel lines are called <i>elements</i> of the cylinder.
Right cylinder	A cylinder with elements perpendicular to the bases. A cylinder that is not a right cylinder is an <i>oblique cylinder</i> .
Prism	A cylinder with polygons for bases. All prisms are special cases of cylinders.
Rectangular prism	A cylinder with rectangles for bases.
Cube	A square prism with square sides.
Cones	
Cone	A solid with exactly one face and a vertex that is not on the face. Straight lines (elements) can be drawn from any point on the edge of the base to the vertex. The base may be any shape at all. The vertex need not be directly over the base.
Circular cone	Cone with a circular base.
Pyramid	Cone with a polygon for a base. All faces joining the vertex are triangles. Pyramids are named by the shape of the base: <i>triangular</i> pyramid, <i>square</i> pyramid, <i>octagonal</i> pyramid, and so on. All pyramids are special cases of cones.

Note that prisms are defined here as a special category of cylinder—a cylinder with a polygon for a base. Figure 7.10 shows a similar grouping of cones and pyramids.

 **Explain the following: Prisms are to cylinders as pyramids are to cones. How is this relationship helpful in learning volume formulas?**

FIGURE 7.9

Cylinders and prisms.



Cylinders have two parallel faces, and parallel lines join corresponding points on these faces. If the parallel faces are polygons, the cylinder can be called a prism.

Special Cylinders

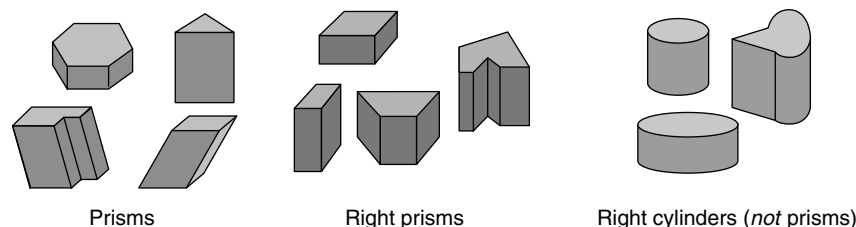
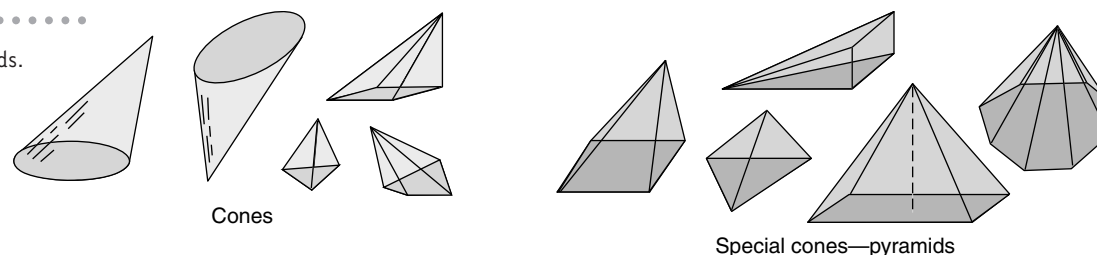
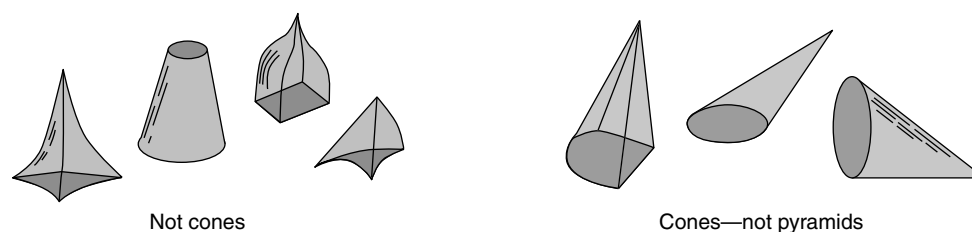


FIGURE 7.10

Cones and pyramids.



Cones and cones with a polygon base (pyramids) all have straight-line elements joining every point of the base with the vertex. (Yes, a pyramid is just a special type of cone.)



Many textbooks define cylinders strictly as circular cylinders. These books do not have special names for other cylinders. Under that definition, the prism is not a special case of a cylinder. This points to the fact that definitions are conventions, and not all conventions are universally agreed upon. If you look at the development of the volume formulas in Chapter 8, you will see that the more inclusive definition of cylinders and cones given there allows one formula for any type of cylinder—hence, prisms—with a similar statement that is true for cones and pyramids.

Sorting and Classifying Activities

The next activity provides a good method when you want to introduce a category of shapes that you plan to define formally later.

ACTIVITY 7.3

Mystery Definition

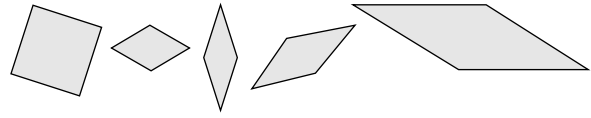
Use the overhead or chalkboard to conduct activities such as the example in Figure 7.11. For your first collection be certain that you have allowed for all possible variables. In Figure 7.11, for example, a square is included in the set of rhombi. Similarly, choose nonexamples to be as close to the positive examples as is necessary to help with an accurate definition. The third or mixed set should also include those nonexamples with which students are most likely to be confused.

Rather than confirm the choice of shapes in the third set, students should write an explanation for their choice.

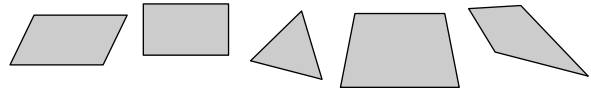
The value of the “Mystery Definition” approach is that students develop ideas and definitions based on their own concept development. After their definitions have been discussed and compared, you can offer the usual “book” definition for the sake of clarity.

For defining types or categories of triangles, the next activity is especially good and uses a different approach.

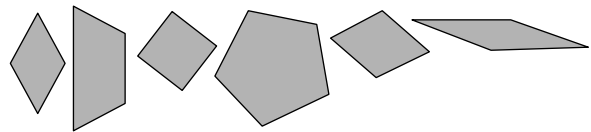
All of these have something in common.



None of these has it.



Which of these have it?



The name of a property is not necessary for it to be understood. It requires more careful observation of properties to discover what shapes have in common.

FIGURE 7.11

All of these, none of these: a mystery definition.

ACTIVITY 7.4

Triangle Sort

Make copies of the Assorted Triangles sheet found in the Blackline Masters. Note the examples of right, acute, and obtuse triangles; examples of equilateral, isosceles, and scalene triangles; and triangles that represent every possible combination of these categories. Have students cut them out. The task is to sort the entire collection into three groups so that no triangle belongs to two groups. When this is done and descriptions of the groupings have been written, students should then find a second criterion for creating three different groupings. Students may need a hint to look only at angle sizes or only at the issue of congruent sides, but hold these hints if you can.

“Triangle Sort” results in definitions of the six different types of triangles without having to list these definitions on the board and have students memorize them. As a follow-up activity, make a chart such as the one shown here. Challenge students to sketch a triangle in each of the nine cells.



BLM 20

	Equilateral	Isosceles	Scalene
Right			
Acute			
Obtuse			



Of the nine cells in the chart, two of them are impossible to fill. Can you tell which ones and why?

Quadrilaterals (polygons with four sides) are an especially rich source of investigations. For the following activity, students should be familiar with the concepts of

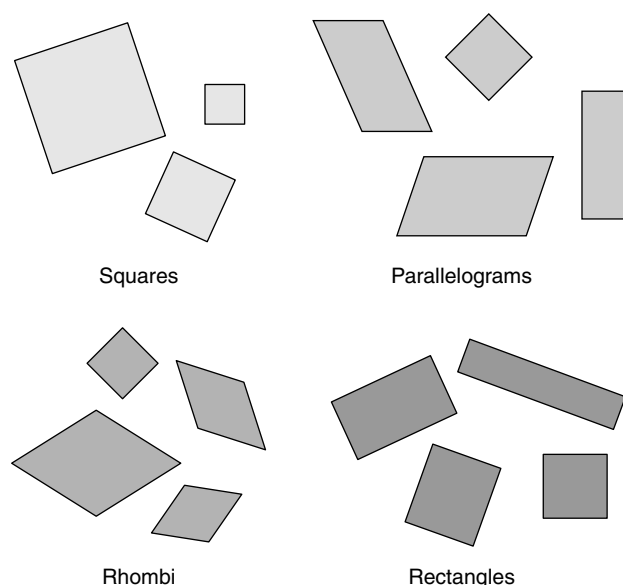


FIGURE 7.12

Shapes for "Property Lists for Quadrilaterals" worksheets can be found in the Blackline Masters.



BLMs 21-24

right, obtuse, and acute angles, congruence of line segments, and symmetry (line and rotational).

ACTIVITY 7.5

Property Lists for Quadrilaterals

Prepare worksheets for parallelograms, rhombi, rectangles, and squares. (See the Blackline Masters.) On each sheet are three or four examples of that category of shape. Examples are illustrated in Figure 7.12. Assign students working in groups of three or four to one type of quadrilateral. Their task is to list as many properties as they can. Each property listed must be applicable to all of the shapes on their sheet. They will need a simple index card to check right angles, to compare side lengths, and to draw straight lines. Mirrors (to check line symmetry) and tracing paper (for angle congruence and rotational symmetry) are also useful tools. Encourage students to use the words "at least" when describing how many of something: for example, "rectangles have at least two lines of symmetry," since squares—included in the rectangles—have four.

Have students prepare their property lists under these headings: Sides, Angles, Diagonals, and Symmetries. Groups then share their lists with the class and eventually a class list for each shape will be developed.

This last activity may take two or three periods. Share lists beginning with parallelograms, then rhombi, then rectangles, and finally squares. Have one group present its list. Then others who worked on the same shape should add to or subtract from it. The class must agree with everything that is put on the list. As new relationships come up in this presentation-and-discussion period, you can introduce proper terminology. For example, if two diagonals intersect in a square corner, then they are *perpendicular*. Other terms such as *parallel*, *congruent*, *bisect*, *midpoint*, and so on can be clarified as you help students write their descriptions. This is also a good time to introduce symbols such as $\cong$ for "congruent" or $\parallel$ for "parallel."

As an extension, repeat Activity 7.5 using kites and trapezoids. "Property Lists for Quadrilaterals" has some important follow-ups that are described in the section on level 2 activities (see p. 203). Furthermore, similar activities can be used to introduce three-dimensional shape definitions.

Construction Activities

Students building or drawing shapes continues to be important at level 1. Dynamic geometry software (*Geometer's Sketchpad*, *The Geometry Inventor*, and *Cabri*) dramatically enhances the exploration of shapes at this level.

In the "Property Lists for Quadrilaterals" activity (Activity 7.5), students examine the diagonals of various classes of quadrilaterals. If that activity has not been done

already, the following exploration is very interesting. Rather than beginning with the shapes, it begins with the diagonals.

ACTIVITY 7.6

Diagonal Strips

For this activity, students need three strips of tagboard about 2 cm wide. Two should be the same length (about 30 cm) and the third somewhat shorter (about 20 cm). Punch nine holes equally spaced along the strip. (Punch a hole near each end. Divide the distance between the holes by 8. This will be the distance between the remaining holes.) Use a brass fastener to join two strips. A quadrilateral is formed by joining the four end holes as shown in Figure 7.13. Provide students with the list of possible relationships for angles, lengths, and ratios of parts. Their task is to use the strips to determine the properties of diagonals that will produce different quadrilaterals. The strips are there to help in the exploration. Students may want to make drawings on dot grids to test the various hypotheses.

Every type of quadrilateral can be uniquely described in terms of its diagonals using only the conditions of length, ratio of parts, and whether or not they are perpendicular. Some students will work with the diagonal relationships to see what shapes can be made. Others will begin with examples of the shapes and observe the diagonal relationships. A dynamic geometry program such as *The Geometer's Sketchpad* is an excellent vehicle for this investigation.

Similar Figures and Proportional Reasoning

A good first definition of similar figures is shapes that “look alike but are different sizes.” More precisely, two figures are *similar* if all of their corresponding angles are congruent and the corresponding sides are proportional. Many proportional reasoning activities are good connections to geometry. Activities 6.8, 6.9, and 6.10 involve scale drawings and proportional relationships in three-dimensional figures that are similar.

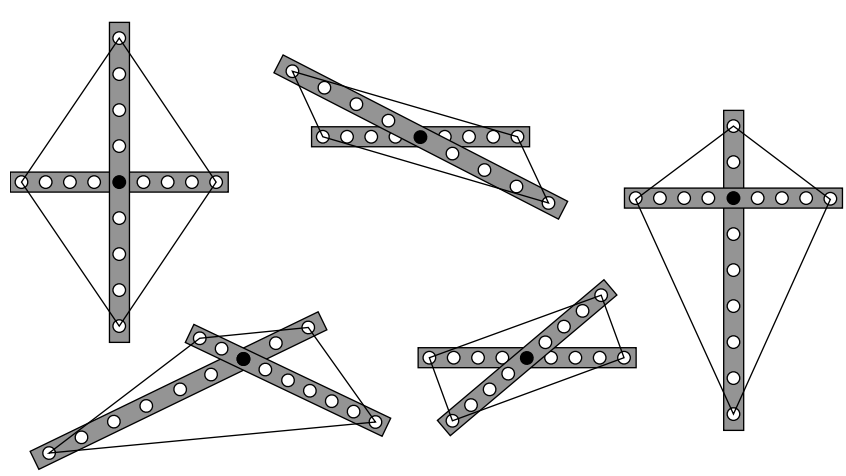
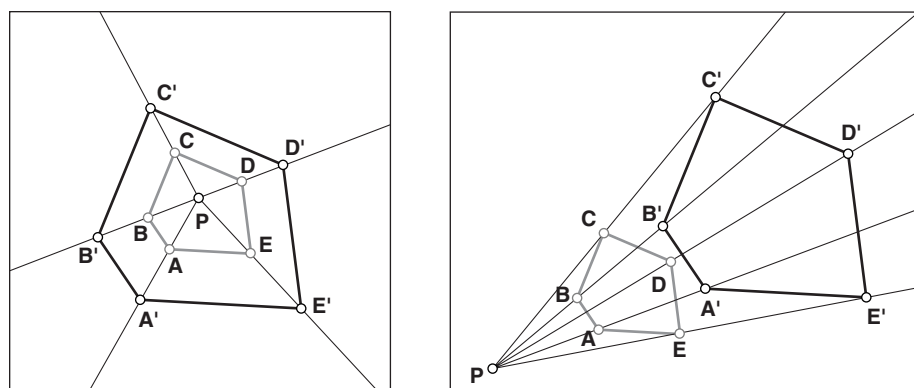


FIGURE 7.13
Diagonals of quadrilaterals.

Quadrilaterals can be determined by their diagonals. Consider the length of each, where they cross, and the angles between them. What conditions will produce parallelograms? Rectangles? Rhombi? Challenge: What properties will produce a nonisosceles trapezoid?

FIGURE 7.14

Begin with figure $ABCDE$ and place point P anywhere at all. Draw lines from P through each vertex. Place point A' twice as far from P as A is from P (scale factor of 2). Do similarly for the other points. In this drawing, $ABCDE$ is the same in both figures and the images are congruent.



A *dilation* is a nonrigid transformation that produces similar figures. Figure 7.14 shows how a given figure can be *dilated* to make larger or smaller figures. If groups of students using different dilation points dilate the same figure using the same scale factor, they will find that the resulting figures are all congruent. Dynamic geometry software makes the results of this exercise quite dramatic. The software allows for the scale factors to be set at any value. Once a dilation is made, the dilation point can be dragged around the screen and the size and shape of the image clearly stay unchanged. Scale factors less than 1 produce smaller figures.

Circles

Many interesting relationships can be observed between measures of different parts of the circle. Among the most astounding and important is the ratio between measures of the circumference and the diameter.



True or False: All circles are similar. Explain.

ACTIVITY 7.7

Discovering Pi

Have groups of students carefully measure the circumference and diameter of many different circles. Each group measures different circles.

Measure both the circumference and diameter of circular items such as jar lids, tubes, cans, and wastebaskets. To measure circumference, wrap string once around the object and then measure that length of string.

Also measure large circles marked on gym floors and playgrounds. Use a trundle wheel or rope to measure the circumference.

Collect measures of circumference and diameter from all groups and enter them in a table. Ratios of the circumference to the diameter should also be computed for each circle. A scatter plot of the data should be made with the horizontal axis representing diameters and the vertical axis circumferences.

Most ratios should be in the neighborhood of 3.1 or 3.2. The scatter plot should approximate a straight line through the origin. The slope of the line should be close to 3.1. (Recall from Chapter 6 that graphs of equivalent ratios are always straight lines through the origin. The exact ratio is an irrational number, about 3.14159, represented by the Greek letter π , pi.)

What is most important in Activity 7.7 is that students develop a clear understanding of π as the ratio of circumference to diameter in any circle. The quantity π is not some strange number that appears in math formulas; it is a naturally occurring and universal ratio.



Technology Note

As students begin to do more than build with geometric “blocks” (tangrams, pattern blocks, grid drawing, etc.), the computer begins to offer powerful tools for explorations. Dynamic geometry software is especially powerful.

Dynamic Geometry Software

In a dynamic geometry program, points, lines, and geometric figures are easily constructed on the computer using only the mouse. Once drawn, the geometric objects can be moved about and manipulated in endless variety. Distances, lengths, areas, angles, slopes, and perimeters can be measured. As the figures are changed, the measurements update instantly.

Lines can be drawn perpendicular or parallel to other lines or segments. Angles and segments can be drawn congruent to other angles and segments. A point can be placed at the midpoint of a segment. A figure can be produced that is a reflection, rotation, or dilation of another figure. The most significant thing is that when a geometric object is created with a particular relationship to another object, that relationship is maintained no matter how either object is moved or changed.

Three of the best-known dynamic geometry programs are *The Geometer's Sketchpad* (Key Curriculum Press, 2001), *Geometry Inventor* (Riverdeep, 1996), and *Cabri Geometry II* (Texas Instruments, 1998). Although each operates somewhat differently, they are sufficiently alike that separate descriptions are not required here. Originally designed for high school students, all can be used profitably and should be used starting about grade 4.

Dynamic Geometry Examples

To appreciate the potential (and the fun) of dynamic geometry software, you really need to experience it on a computer. In the meantime, an example is offered here in an attempt to illustrate how these programs work.

In Figure 7.15, the midpoints of a freely drawn quadrilateral ABCD have been joined. The diagonals of the resulting quadrilateral (EFGH) are also drawn and measured. No matter how the points A, B, C, and D are dragged around the screen, even inverting the quadrilateral, the other lines will maintain the same relationships (joining midpoints and diagonals), and the measurements will be instantly updated on the screen.

Remember that at level 1, the objects of thought are *classes* of shapes. In a dynamic geometry program, if a quadrilateral is drawn, only one shape is observed, as would be the case on paper or on a geoboard. But now that quadrilateral can be stretched and altered in endless ways. Students actually explore not one shape but an enormous number of examples from that class of shapes. If a property does not change when the figure changes, the property is attributable to the *class* of shapes rather than any particular shape.

Another example in Figure 7.16 shows how *Sketchpad* can be used to investigate quadrilaterals starting with the diagonals. The directions for creating the sketch are

FIGURE 7.15

A *Sketchpad* construction illustrating an interesting property of quadrilaterals.

Distance (J to E) = 3.81 cm
Distance (G to J) = 3.81 cm
Distance (J to H) = 2.21 cm
Distance (F to J) = 2.21 cm

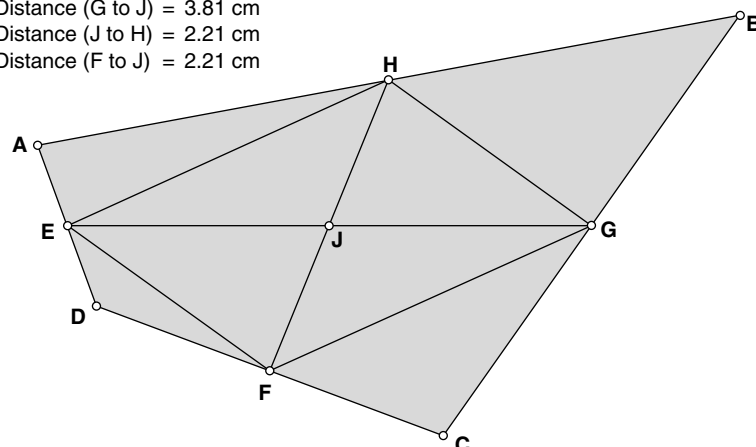
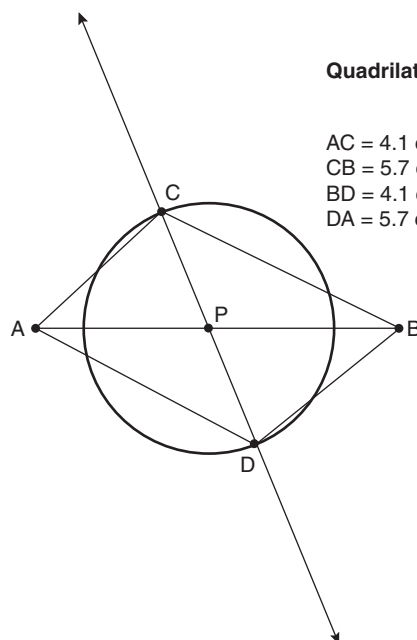


FIGURE 7.16

With *The Geometer's Sketchpad* students can construct two line segments that will always bisect each other. When the endpoints are joined, the resulting quadrilateral will always be of the same class, regardless of how points A, B, C, and D are moved around.

Quadrilaterals with diagonals that bisect each other.

AC = 4.1 cm
CB = 5.7 cm
BD = 4.1 cm
DA = 5.7 cm



Draw segment AB with midpoint P.
Construct circle with center P and control point C.
Construct line through C and P and then construct intersection point D.
Construct ABCD and measure each side.
Drag C. What different quadrilaterals can you make?
What is true about the diagonals of every shape you make?
What can you find out about the diagonals of the shapes you make?

included and can be done quite simply with minimal experience with the software. By creating the drawing in this manner, the diagonals of ABCD will always bisect each other no matter how the drawing is altered. By dragging point C around, ABCD can be made into a parallelogram, rectangle, rhombus, and square. But for each of these figures, additional information about the diagonals can be determined by looking at the drawing.

Dynamic geometry programs are also powerful for investigating concepts of symmetry and transformations (slides, flips, and turns). The publishers of these programs provide excellent activities that are appropriate for level 1 investigations. Many activities are included with the software, and others are found in supplemental publications.



Why can't the drawing in Figure 7.16 be transformed into a kite or a trapezoid that is not also a parallelogram?

The symmetry investigations (both line symmetry and rotational symmetry), “Diagonal Strips” (Activity 7.6), investigations of similarity, and “Discovering Pi” (Activity 7.7) can all be explored profitably with dynamic geometry programs. The publishers of these programs provide excellent activities that are appropriate for level 1 investigations. Many activities are included with the software, and others are found in supplementary publications.

Activities for Level 2 Thinkers

The hallmark of level 2 activities is the inclusion of informal logical reasoning. Most fifth- to eighth-grade students will still be at level 1 at best. However, as students develop an understanding of various geometric properties and attach these properties to important categories of shapes, it is essential to begin to encourage conjecture and to explore informal deductive arguments. Do not be afraid to explore some of the activities at this level just because you may be teaching fifth or sixth grade.

Definitions and Proofs

To really understand the difference between levels 1 and 2 of the van Hiele theory, contrast the required thinking in the level 1 activity “Property Lists for Quadrilaterals” (Activity 7.5) and the following activity that is designed as a follow-up to that one.

ACTIVITY 7.8

Minimal Defining Lists

(This activity must be done as a follow-up to the “Property Lists for Quadrilaterals” activity on p. 198.) Once property lists for the parallelogram, rhombus, rectangle, and square (and possibly the kite and trapezoid) have been agreed on by the class, have these lists posted or type them up and duplicate them. In groups, the task is to find “minimal defining lists,” or MDLs, for each shape. An MDL is a subset of the properties for a shape that is “defining” and “minimal.” “Defining” here means that any shape that has all the properties on the MDL *must* be that shape. Thus, an MDL for a square will guarantee that you have a square. “Minimal” means that if any single property is removed from the list it is no longer defining. For example, one MDL for a square is a quadrilateral with four congruent sides and one right angle. Students should attempt to find at least two or three MDLs for their shape. A proposed list can be challenged as either not minimal or not defining. A list is not minimal if a property can be removed yet the list still defines the shape. A list is not defining if a counterexample—a shape other than one being described—can be produced using only the properties on the list.

The parallelogram, rhombus, rectangle, and square each have at least four MDLs. One of the most interesting MDLs for each shape consists only of the properties of its diagonals. For example, a quadrilateral with diagonals that bisect each other and are perpendicular (intersect at right angles) is a rhombus. Several MDLs have only one property. For example, a parallelogram is a quadrilateral with rotational symmetry of at least order 2.

The MDL activity is worth some further discussion. First, notice the logic component. “If a quadrilateral has these properties, *then* it must be a square.” Logic is also involved in disproving a faulty list. A second feature is the opportunity to discuss what constitutes a definition. In fact, any MDL could be the definition of the shape. The definitions we usually use are MDLs that have been chosen probably due to the ease with which we can understand them. A quadrilateral with diagonals that bisect each other does not immediately call to mind a parallelogram. Recall that when students created their property lists, no definition was given, only a collection of shapes and a label. Theoretically, the lists could have been created without ever having heard of these shapes. Finally, notice that the object of students’ thinking in this activity is clearly on properties, not on shapes. The products of the activity are relationships among the properties.

The next activity is also a good follow-up to the “Property Lists for Quadrilaterals” activity, although it is not restricted to quadrilaterals and can include three-dimensional shapes as well. Notice again the logic involved.

ACTIVITY 7.9

True or False?

Prepare statements of the following forms: “If it is a _____, then it is also a _____.” “All _____ are _____.” “Some _____ are _____.”

A few examples are suggested here but numerous possibilities exist.

- If it is a square, then it is a rhombus.
- All squares are rectangles.
- Some parallelograms are rectangles.
- All parallelograms have congruent diagonals.
- If it has exactly two lines of symmetry, it must be a quadrilateral.
- If it is a cylinder, then it is a prism.
- All prisms have a plane of symmetry.
- All pyramids have square bases.
- If a prism has a plane of symmetry, then it is a right prism.

The task is to decide if the statements are true or false and to present an argument to support the decision. Four or five true-or-false statements will make a good lesson. Once this format is understood, let students challenge their classmates by making their own lists of five statements. Each list should have at least one true statement and one false statement. Use the students’ lists in subsequent lessons.



Use the property list for squares and rectangles to prove “All squares are rectangles.” Notice that you must use logical reasoning to understand this statement. It does little good to simply force it on students who are not ready to develop the relationship.

Although logic has been involved in the previous activities, you may have difficulty understanding how middle school students can actually do proofs. The following activity was designed by Sconyers (1995) to demonstrate that students can create proofs in geometry well before high school.

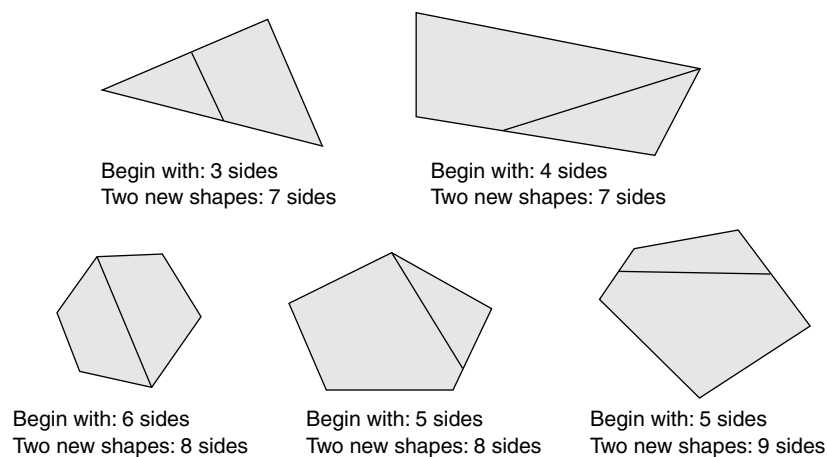


FIGURE 7.17 Start with a polygon and draw a segment to divide it into two polygons. How many sides will the two new polygons have?

ACTIVITY 7.10

Two Polygons from One

Pose the following problem:

Begin with a convex polygon with a given number of sides. Connect two points on the polygon with a line segment forming two new polygons. How many sides do the two resulting polygons have together?

Demonstrate with a few examples (see Figure 7.17). Have students explore by drawing polygons and slicing them. Encourage students to make a table showing sides in original and resulting sides. Students should first make conjectures about a general rule. When groups are comfortable with their conjectures, they should try to reason why their statement is correct—that is, prove their conjectures.

Obviously, the number of resulting sides depends on where the slice is made. With the exception of triangles, there are three possibilities. For each case, a clear argument can be made. The appropriate conjecture and proof are left to you, but trust that students working together can do this task.

Notice that in this task, as in others we have explored, the statements to be proved come from students. If you write a theorem on the board and ask students to prove it, you have already told them that it is true. If, by contrast, a student makes a statement about a geometric situation the class is exploring, it can be written on the board with a question mark as a *conjecture*, a statement whose truth has not yet been determined. You can ask, “Is it true? Always? Can we prove it? Can we find a counterexample?” Reasonable deductive arguments can be forged out of discussions.

The Pythagorean Relationship

The *Pythagorean relationship* is so important that it deserves special attention. In geometric terms, this relationship states that if a square is constructed on each side of a right triangle, the areas of the two smaller squares will together equal the area of the square on the longest side, the hypotenuse. To discover this relationship, consider the following activity.

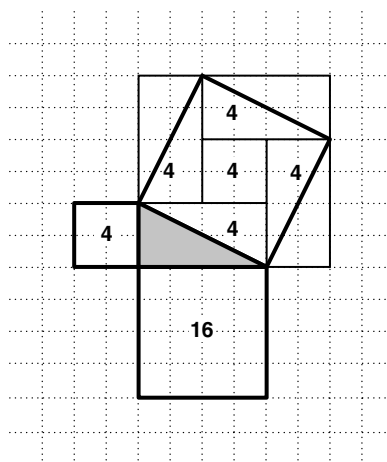


FIGURE 7.18

The Pythagorean relationship. Note that if drawn on a grid, the area of all squares is easily determined. Here $4 + 16 =$ area of the square on the hypotenuse.

ACTIVITY 7.11

The Pythagorean Relationship

Have students draw a right triangle on half-centimeter grid paper. Assign each student a different triangle by specifying the lengths of the two legs. Students are to draw a square on each leg and the hypotenuse and find the area of all three squares. (For the square on the hypotenuse, the exact area can be found by making each of the sides the diagonal of a rectangle. See Figure 7.18.) Make a table of the area data (Sq. on leg 1, Sq. on leg 2, Sq. on hyp.) and ask students to look for a relationship between the squares.

As an extension to the last activity, students can explore drawing other figures on the legs of right triangles and computing areas. For example, they can draw semicircles or equilateral triangles instead of squares. The areas of any regular polygons drawn on the three sides of right triangles will have the same relationship.

Activity 7.11 establishes the Pythagorean relationship. What about a proof? Figure 7.19 shows two proofs that students can follow. The first consists of only the two drawings. It is taken from the book *Proofs without Words* (Nelson, 1993). An algebraic proof is shown below the drawings, based on the second square.



Use the two drawings in Figure 7.19 to create a proof of the Pythagorean relationship.

Technology Note

The *e-Standards* includes a dynamic proof without words of the Pythagorean relationship that is worth sharing with your students (Applet 6.5). Because it requires knowing that parallelograms and rectangles with the same base and height have the same area (see Chapter 8), it is also a good review.



Finding Versus Explaining Relationships

Dynamic geometry software such as *The Geometer's Sketchpad* allows students to explore an entire class of figures and observe properties or relationships that are attributable to that class. At level 2, however, the focus is on reasoning or deductive thinking. Can these computer programs help students develop deductive arguments to support the relationships they come to believe through inductive reasoning? Consider the following situation.

Suppose that you have students use a dynamic geometry program to draw a triangle, measure all of the angles, and add them up. As the triangle vertices are dragged around, the sum of the angles would remain steadfast at 180 degrees. Students can conjecture that the sum of the interior angles of a triangle is always 180 degrees, and they would be completely convinced of the truth of this conjecture based on this inductive experience. (Several noncomputer activities lead to the same conclusion.)

As Michael de Villiers notes in his excellent book *Rethinking Proof with the Geometer's Sketchpad* (1999), "The observation that the sun rises every morning does not explain why this is true" (p. 24). De Villiers points out that the experience leading to the conjecture or truth should also help students develop a rationale for the result. In the case of interior angles of a triangle, the experience just described fails to explain *why it is so*. Consider the following activity, which can be done easily with paper and scissors or quite dramatically with a dynamic geometry program.

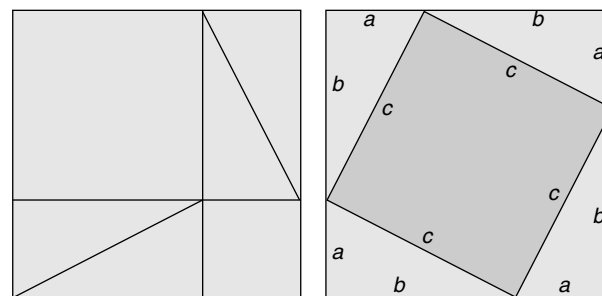
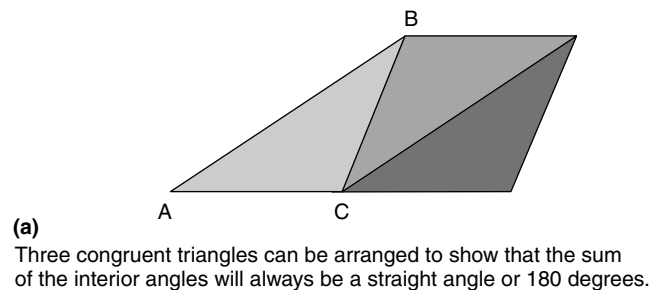
ACTIVITY 7.12

Angle Sum in a Triangle

Have all students cut out three congruent triangles. (Stack three sheets of paper and cut three shapes at one time.) Place one triangle on a line and the second directly next to it in the same orientation. Place the third triangle in the space between the triangles as shown in Figure 7.20(a). Based on this experience, what conjecture can you make about the sum of the angles in a triangle?

In a dynamic geometry program, the three triangles in Figure 7.20(a) can be drawn by starting with one triangle, translating it to the right the length of AC, and then rotating the same triangle about the midpoint of side BC. When vertices of the original triangle are dragged, the other triangles will change accordingly and remain congruent. We still do not know why the angle sum is always a straight angle, but this exploration allows students to see why it might be so. In the figure, there are lines parallel to each side of the original triangle. By using properties of angles formed by cutting parallel lines with a transverse line, it is easy to argue that the sum of the angles will always be a straight line (see Figure 7.20(b); the proof is left to you).

Dynamic geometry software can be enormously powerful for helping students observe geometric relationships and make conjectures. The truth of the conjectures will often be obvious. At level 2, however, we must begin to ask why. The following activity further illustrates the point.



The area of the large square is $(a + b)^2 = a^2 + 2ab + b^2$

The same area is also c^2 plus 4 times the area of one triangle.

$$c^2 + 4 \left(\frac{1}{2} ab \right) = c^2 + 2ab$$

So $c^2 + 2ab = a^2 + 2ab + b^2$

$$c^2 = a^2 + b^2$$

FIGURE 7.19

Two proofs of the Pythagorean relationship. The two squares together are a "proof without words." Can you supply the words? The second proof is the algebraic proof based on the right-hand figure.

FIGURE 7.20

Deductive, logical reasoning is necessary to *prove* relationships that appear true from observations.

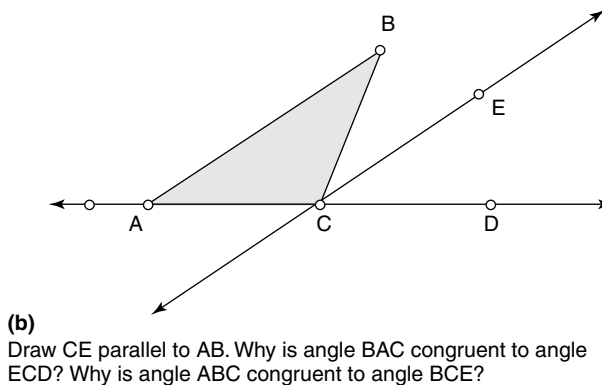
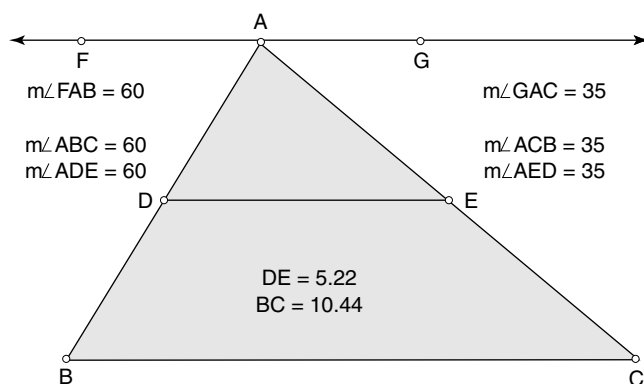


FIGURE 7.21

The midsegment of a triangle is always parallel to the base and half as long.



EXPANDED LESSON

(pages 228–229)

A complete lesson plan based on “Triangle Midsegments” can be found at the end of this chapter.

ACTIVITY 7.13

Triangle Midsegments

Using a dynamic geometry program, draw a triangle, and label the vertices A, B, and C. Draw the segment joining the midpoints of AB and AC, and label this segment DE, as in Figure 7.21. Measure the length of DE and BC. Also measure angles ADE and ABC. Drag points A, B, and C. What conjectures can you make about the relationships between segment DE, the *midsegment* of ABC, and BC, the base of ABC?

It is very clear that the midsegment is half the length of the base and parallel to it, but why is this so? Students will need a bit more guidance, but you should not necessarily have to provide the argument for them. Suggest that they draw a line through A parallel to BC. List all pairs of angles that they know are congruent. Why are they congruent? Note that triangle ABC is similar to triangle ADE. Why is it similar? With hints such as these, many middle grade students can begin to make logical arguments for why the things they observe to be true are in fact true.

Assessment Note

With a greater focus on explaining *why* something is true, we need to consider how students may approach this endeavor. Researchers hypothesize that students’ abilities to justify and prove follow a developmental path that progresses from inductive to deductive approaches.



Stage 0: Students are ignorant of the need for, or existence of, proof.

Stage 1: Students are aware of the notion of proof but consider checking a few cases as sufficient.

Stage 2: Students are aware that checking a few cases is not sufficient but are satisfied that either (a) checking extreme cases or random cases is proof, or (b) use of a generic example forms a proof for a class of objects.

Stage 3: Students are aware of the need for a general argument but are unable to produce such arguments themselves; however, they are likely to be able to understand the generation of such an argument (for example, by a classmate). This also includes the ability to follow

a short chain of deductive reasoning. At this stage, students are able to apply a definition without allowing intuitive and familiar notions of a concept to interfere with the application of the definition. They are also able to understand necessary and sufficient conditions.

Stage 4: Students are aware of the need for a general argument, are able to understand the generation of such an argument, and are able to produce such arguments themselves in a limited number of (familiar) contexts.

Stage 5: Students are aware of the need for a general argument, are able to understand the generation of such an argument (including more formal arguments), and are able to produce such arguments themselves in a variety of contexts (both familiar and unfamiliar) (Knuth, Choppin, Slaughter, & Sutherland, 2002, pp. 1694–1695).

Knuth et al.(2002) suggest that middle school students overwhelmingly rely on the use of examples as a method of demonstrating and/or confirming the truth of a statement (Stage 1/Stage 2). It is very likely that most students in grades 5 to 8 will be functioning between stages 0 and 3. As you work with students to help them generate logical arguments for why things are true, keep in mind these stages of proof concept development to help you make sense of students' approaches.

Transformation Activities

Transformations are also called “rigid motions”—movements that do not change the size or shape of the object moved. Usually, three transformations are discussed in grades 5–8: *translations* or slides, *reflections* or flips, and *rotations* or turns. The goal of instruction is to help students recognize and apply these transformations. You can use a nonsymmetrical shape on the overhead to introduce these terms (see Figure 7.22). Most elementary school textbooks only use the center of the figure as the point of rotation and a vertical or horizontal line as the line of reflection. These restrictions are for simplicity. In general, the center of rotation can be anywhere on or off the figure. Lines of reflection can also be anywhere.

Interestingly, the study of symmetry is also included under the study of transformations. Do you know why?

If a shape can be folded on a line so that the two halves match, then it is said to have *line symmetry* (or mirror symmetry). Notice that the fold line is actually a line of reflection—the portion of the shape on one side of the line is reflected onto the other side. That is the connection between line symmetry and transformations.

To explain line symmetry to students, you can show examples and nonexamples. Another approach is to have students fold a sheet of paper in half and cut out a shape of their choosing. When they open the paper, the fold line will be a line of symmetry. A third way is to use mirrors. When you place a mirror on a picture or design so that

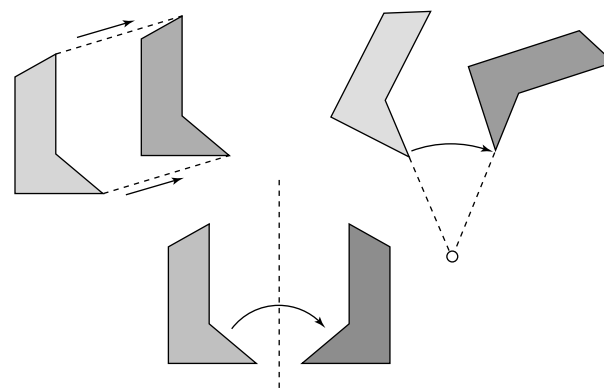


FIGURE 7.22 Translation (slide), reflection (flip), rotation (turn).

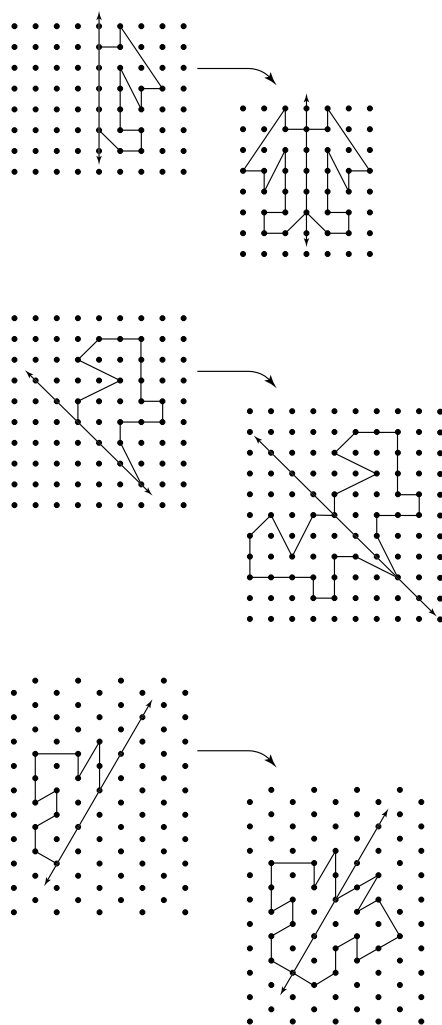


FIGURE 7.23
Exploring symmetry on dot grids.

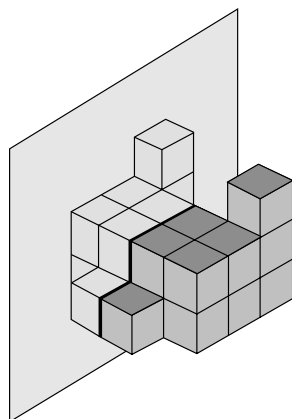


FIGURE 7.24
A block building with one plane of symmetry.

the mirror is perpendicular to the table, you see a shape with symmetry when you look in the mirror.

Here is an activity with line symmetry.

ACTIVITY 7.14

Dot Grid Line Symmetry

For this activity, students need to use either isometric or rectangular dot grid paper. Students should draw a line through several dots. This line can be horizontal, vertical, or skewed. Students should make a design completely on one side of the drawn line that touches the line in some way (see the left-hand drawings in Figure 7.23). Now the task is to make the mirror image of their design on the other side of the line. (Students can exchange designs and make the mirror image of a classmate's design.) When finished, they can use a mirror to check their work. They place the mirror on the line and look into it from the side of the original design. With the mirror in place, they should see exactly the same image as they see when they lift the mirror. You can also challenge them to make designs with more than one line of symmetry.

A plane of symmetry in three dimensions is analogous to a line of symmetry in two dimensions. Figure 7.24 illustrates a shape built with cubes that has a plane of symmetry.

ACTIVITY 7.15

Plane Symmetry Buildings

With cubes, build a building that has a plane of symmetry. If the plane of symmetry goes between cubes, slice the shape by separating the building into two symmetrical parts. Try making buildings with two or three planes of symmetry. Build various prisms. Do not forget that a plane can slice diagonally through the blocks.

A shape has *rotational symmetry* if it can be rotated about a point and land in a position exactly matching the one in which it began. A square has rotational symmetry as does an equilateral triangle.

A good way to understand rotational symmetry is to take a shape with rotational symmetry, such as a square, and trace around it on a piece of paper. Call this tracing the shape's "box." The order of rotational symmetry will be the number of ways that the shape can fit into its box without flipping it over. A square has rotational symmetry of *order 4*, whereas an equilateral triangle has rotational symmetry of *order 3*. The parallelogram in Figure 7.25 has rotational symmetry of *order 2*. Some books would call order 2 symmetry "180-degree symmetry." The number of degrees refers to the smallest angle of rotation required before the shape matches itself or fits into its box. A square has 90-degree rotational symmetry.

ACTIVITY 7.16

Pattern Block Rotational Symmetry

Have students construct designs with pattern blocks with different rotational symmetries. They should be able to make designs with order 2, 3, 4, 6, or 12 rotational symmetry. Which of the designs have mirror symmetry as well?

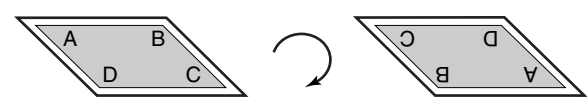


FIGURE 7.25

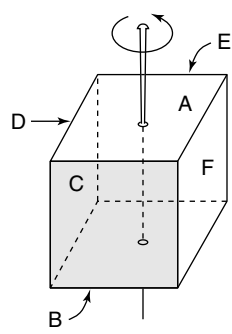
This parallelogram fits in its box two ways without flipping it over. Therefore, it has rotational symmetry of order 2.

Rotational symmetry in the plane (also referred to as *point symmetry*) has an analogous counterpart in three dimensions. Whereas a figure in a plane is rotated about a point, a three-dimensional figure is rotated about a line. This line is called an *axis of symmetry*. As a solid with rotational symmetry revolves around an axis of symmetry, it will occupy the same position in space (its “box”) but in different orientations. A solid can have more than one axis of rotation. For each axis of symmetry, there is a corresponding order of rotational symmetry. A regular square pyramid has only one axis of symmetry that runs through the tip of the pyramid and the center of the square. A cube, by contrast, has a total of 13 axes of symmetry: three (through opposite faces) of order 4, four (through diagonally opposite vertices) of order 3, and six (through midpoints of diagonally opposite edges) of order 2.

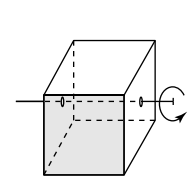
ACTIVITY 7.17

Find the Spin Lines

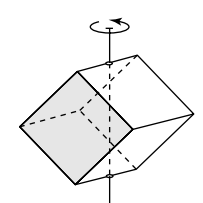
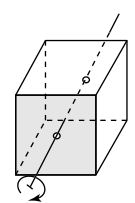
Give students a solid shape that has one or more axes of rotational symmetry. Color or label each face of the solid to help keep track. The task is to find all axes of rotational symmetry (spin lines) and determine the order of rotational symmetry for each. Suggest that students use one finger of each hand to hold the solid at the two points where the axis of symmetry emerges. A partner can then slowly turn the solid, and both can decide when the solid is again “in its box”—that is, in the same space it was in originally (see Figure 7.26).



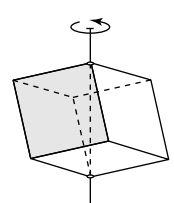
With A on top, the cube fits in its “box” four ways. Through this axis, the order of rotational symmetry is order 4.



These two axes also have rotational symmetry of order 4.



Edge-to-edge axes each have symmetry of order 2. How many are there?



If the axis is corner to opposite corner, what is the order of symmetry? How many of these axes are there?

FIGURE 7.26

Rotations of a cube.

Composition of Transformations

One transformation can be followed by another. For example, a figure can be reflected over a line, and then that figure can be rotated about a point. A combination of two or more transformations is called a *composition*.

Have students experiment with compositions of two or even three transformations using a simple shape on a rectangular dot grid. For example, have students draw an L-shape on a dot grid and label it L_1 . (Refer to Figure 7.27.) Reflect it through a line, and then rotate the image $\frac{1}{4}$ turn clockwise about a point not on the shape. Call this image L_2 , the image of a composition of a reflection followed by a rotation. Notice that if L_1 is rotated $\frac{1}{4}$ turn clockwise about the same point used before to L_3 there is a relationship between L_2 and L_3 . (How could you get L_3 from L_2 ?)



A composite transformation can involve more than two individual transformations, just as multiplication can involve more than one factor (e.g., $3 \times 8 \times 2$). Can you describe a composite transformation to get from L_1 to L_3 that is the same as the single transformation of a quarter turn about A?

Continue to explore different combinations of transformations. Don't forget to include translations (slides) in the compositions. Compositions do not have to involve different types of transformations. For example, a reflection can be followed by another reflection. When your students understand compositions of transformations on a grid, challenge them with the next task.

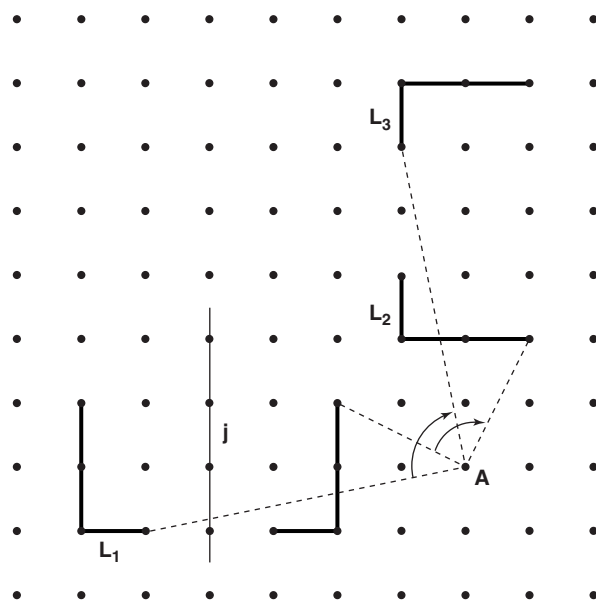


FIGURE 7.27

Shape L_1 was reflected across line j and rotated $\frac{1}{4}$ turn about point A resulting in L_2 . L_1 was also rotated $\frac{1}{4}$ turn about point A . How are L_2 and L_3 related? Will this always work?

ACTIVITY 7.18

Mystery Transformations

Draw a small L-shaped figure near one corner of a rectangular dot grid. On this page, students draw a congruent L-shape somewhere near the center of the page. The second L can be flipped or turned in any orientation they wish. Then students trade papers with a partner. The task is to find some combination of slides, flips, and $\frac{1}{4}$ or $\frac{1}{2}$ turns that will take the shape in the corner onto the shape drawn by the students.

This is a challenging activity. To help students, ask if the L has been flipped over. If it is flipped over, then there will have to be one or three reflections. Regardless of how the two shapes are oriented, it can always be done in three or fewer transformations.

Technology Note

In NCTM's *e-Standards*, "Understanding Congruence, Similarity, and Symmetry" (Applet 6.4)



is one of the best examples of a simple yet valuable interactive applet. In the first part of the applet, students develop an understanding of all three rigid motions. In the second part, a transformation is complete and the student uses a guess-and-check procedure to determine which exact transformation was done. In the last two parts, students can explore compositions of reflections and then other compositions up to three transformations. This applet is strongly recommended.

Tessellations Revisited

Either by using transformations or by combining compatible polygons, students can create tessellations that are artistic and quite complex.

The Dutch artist M. C. Escher is well known for his tessellations, where the tiles are very intricate and often take the shape of things like birds, horses, angels, or lizards. Escher took a simple shape, such as a triangle, parallelogram, or hexagon, and performed transformations on the sides. For example, a curve drawn along one side might be translated (slid) to the opposite side. Another idea was to draw a curve from the midpoint of a side to the adjoining vertex. This curve was then rotated about the midpoint to form a totally new side of the tile. These two ideas are illustrated in part (a) of Figure 7.28. Dot paper is used to help draw the lines. *Escher-type tessellations*, as these have come to be called, are quite popular projects for students in grades 5 and up. Once a tile has been designed, it can be cut from two different colors of construction paper instead of drawing the tessellation on a dot grid.

A *regular tessellation* is made of a single tile that is a regular polygon (all sides and angles congruent). Each vertex of a regular tessellation has the same number of tiles meeting at that point. A checkerboard is a simple example of a regular tessellation. A *semiregular tessellation* is made of two or more tiles, each of which is a regular polygon. At each vertex of a semiregular tessellation, the same collection of regular polygons comes together in the same order. A vertex (and, therefore, the complete semiregular tessellation) can be described by the series of shapes meeting at a vertex. Under each example of these tessellations in part (b) of Figure 7.28, the vertex numbers are given. Students can figure out what polygons are possible at a vertex and design their own semiregular tessellations.

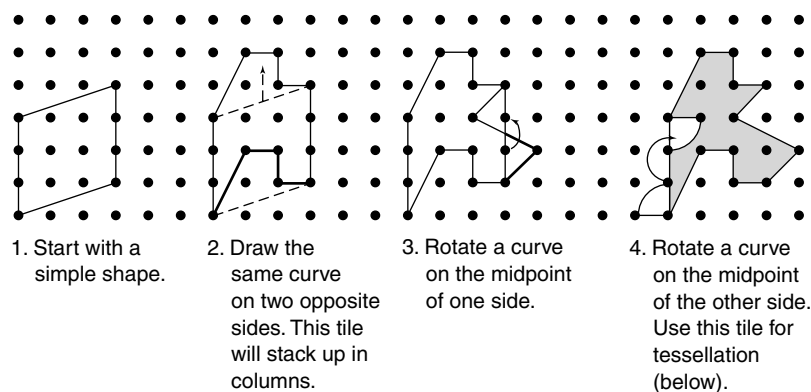
Tessellations like these are very popular with teachers. A number of excellent resource books can be found in catalogs such as ETA/Cuisenaire.



Technology Note

Several computer programs make it easy to create Escher-type tessellations. One example is *TesselMania!* (The Learning Company, 1994).

Although exciting to use, these programs should be used with care. It is so easy to create intricate results with almost no understanding of the transformations being used. Planning on paper before using the computer is strongly recommended. Then the computer can be used to do the tedious work of creating a finished product complete with color and embellishments.



A column of this tile will now match a like column that is rotated one complete turn. Find these rotated columns in the tessellation below.

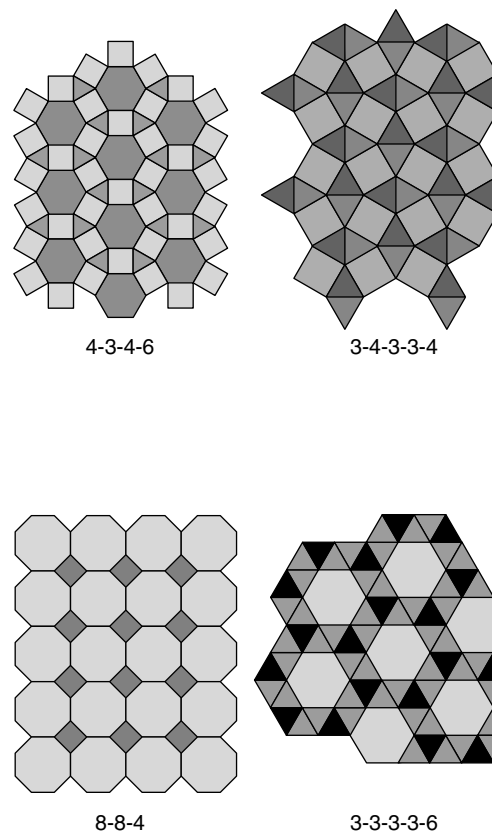
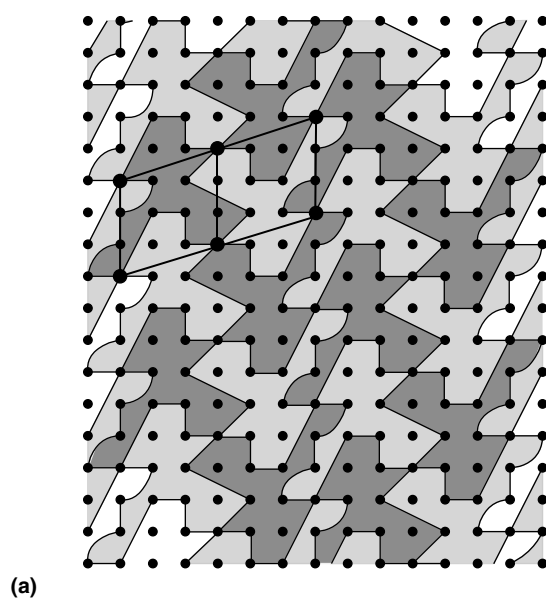


FIGURE 7.28

(a) One of many ways to create an Escher-type tessellation. (b) Examples of semiregular tessellations.

The following activity is a challenge to establish an interesting relationship between symmetries and transformations. The activity utilizes the 12 pentomino shapes, the set of all shapes made from five squares joined at matching edges. Many students in the upper grades will have done pentomino activities in earlier grades. If your students are not familiar with pentominoes, you may want to have them do the “Pentominoes” activity (Activity 7.26, p. 222) in the Visualization section that has students develop them.

ACTIVITY 7.19

Pentomino Positions

Have students cut out a set of 12 pentominoes from 2-cm grid paper. (See Figure 7.29.) Mark one side of each piece to help remember if it has been flipped over. The first part of the task is to determine how many different positions on the grid each piece has. Call positions “different” if a reflection or a turn is required to make them match. Therefore, the cross-shaped piece has only

one position. The strip of five squares has two positions. Some pieces have as many as eight positions. The second part of the task is to find a relationship between the line symmetries and rotational symmetries for each piece and the number of positions it can have on the grid.

The first part of this activity was adapted from the classic book *Boxes, Squares, and Other Things* (Walter, 1970), which has many more excellent explorations. The second part of the problem is not hard if students make a table of what they know. The solutions to the task are left to you and your students. A follow-up to this activity can be found in *Boxes, Squares, and Other Things*.

The next task challenges students to develop generalizations about certain compositions of transformations.

ACTIVITY 7.20

Double Reflections

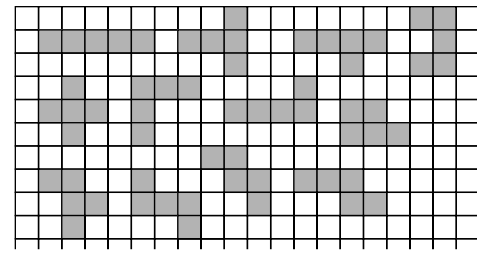
There are two similar tasks in this activity:

1. Verify that the composition of two reflections across parallel lines (first across one and then the other) is the same as a translation. Make a drawing on a dot grid to illustrate. Challenge: If you know the distance between the two parallel lines, what can you say about the translation—both its direction and its distance? Does it matter where the shape begins (between the lines or outside) or which line is used as the first reflection? Explain your conclusions.
2. Verify that the composition of two reflections across intersecting lines is the same as a rotation about the point of intersection of the lines. Make a drawing on a dot grid to illustrate. Challenge: How is the angle of rotation related to the angle between the lines? Explain your conclusions.

In the first problem, the translation distance is always double the distance between the lines and is perpendicular to them. The direction is determined by which line is used for the first reflection in relation to the preimage. In the second problem, the angle of rotation is double the angle between the lines. It is possible to construct reasonable arguments for these conclusions by examining drawings.

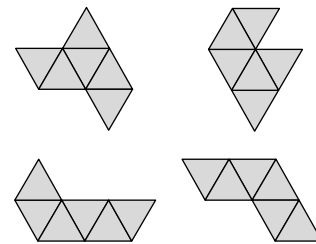
Location Activities

Location activities prior to this level should prepare students for coordinates and build on students' position language (*over, under, near, far, between, left, and right*). The need for a coordinate system should be developed if it has not been already. Through location activities, students in grades 5–8 should become more familiar with the four

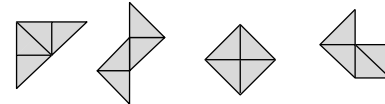


There are 12 pentominoes.

Finding all possible shapes made with five squares—or six squares (called “hexominoes”) or six equilateral triangles and so on—is a good exercise in spatial problem solving.



Four of the different shapes that six equilateral triangles will make.



Four of the different shapes that four “half-square” triangles will make.

FIGURE 7.29

Pentominoes and related shape challenges.

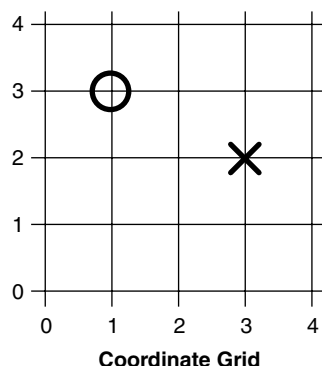


FIGURE 7.30
A simple coordinate grid. The X is at (3, 2) and the O is at (1, 3). Use the grid to play Three in a Row (like Tic-Tac-Toe). Put marks on intersections, not spaces.

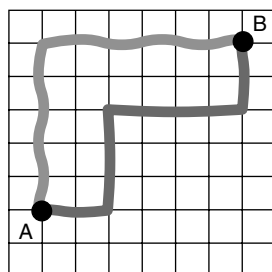


FIGURE 7.31
Coloring in different paths on a grid. What is the fewest number of turns needed to get from A to B? The most?

quadrants in the coordinate grid. However, if your students are not familiar with grids and coordinates, begin with activities that use the first quadrant only.

Once a coordinate system has been introduced, students may want to use it in a simple game similar to the commercial game called “Battleship.” Each player has a grid similar to the one in Figure 7.30. Players secretly put their initials on five intersections of their own grid. Then, with the grids kept hidden from each other, the players take turns trying to “hit” the other player’s targets by naming a point on the grid using coordinates. The other player indicates if the “shot” was a hit or a miss. When a player scores a hit, he or she gets another turn. Each player keeps track of where he or she has taken shots, recording an “X” for a hit and an “O” for a miss. The game ends when one player has hit all of the other player’s targets.

At this level, students are simply using coordinates to describe positions. Although this is not very challenging, it provides practice for finding coordinates in a gamelike setting that can be motivating for students.

Taxicab Geometry: A Non-Euclidean Excursion

When determining coordinates, many students become confused about what they are supposed to count—is it the interval between points or is it the points? In taxicab geometry you are allowed to “travel” only on grid lines, as if you are traveling in a taxi-cab on the streets of a city laid out in grid form. Like taxis, travel is always toward the destination. If moving toward the upper right, all paths are either upward or to the right. Distance between points is determined by counting the intervals (city blocks) between intersections. Figure 7.31 illustrates two of many legitimate paths between points A and B. Either path is a distance of 11 units. Will paths between two points always be the same length?

The next activity is an introduction to this different version of geometry.

ACTIVITY 7.21

Taxicab Geometry Distances

Explain the following scenario to students:

Katie and Clarissa want to meet at a restaurant that is equidistant from each of their houses and such that the total of the distances they each travel does not exceed 8 miles. Their city is arranged like a grid, and they can either drive or walk along the streets, but they cannot cut diagonally across a block. One block is 1 mile. If Clarissa’s house is 4 miles east and 2 miles north of Katie’s, draw a map showing where they could look for a restaurant. Explain.

It may be helpful to suggest that students locate Katie’s house at (0, 0), although this is not necessary.



Before continuing, try “Taxicab Geometry Distances.” What patterns do you see? Look both at the coordinates and at the visual image you get.

In solving the problem in Activity 7.21, it is likely that students will first find the points that are 3 miles from each house since the distance between the houses is 6.

Interestingly, there are three such points in this problem. But then you have to look for a longer distance for each to travel. There are lots of points that are 4 miles from each house but only two that are 4 miles from both houses and similarly for 5, 6, 7, and 8 miles away. If Katie's house is placed at $(0, 0)$, the sum of the coordinates forms a pattern. The closest three points have a sum of 3. The points at 4 miles have coordinates of $(1, 3)$ and $(3, -1)$. If you add absolute values, these sums are 4. The sum for the next two points is 5, and so on. If Katie's house is placed at a point other than $(0, 0)$, there will still be a pattern, but it will be a bit different. In your discussion with students, try to get them to see most of these ideas.

When looking for points farther away from each house, say, 5 miles from each, it may be useful to look to find *all* of the points that are 5 miles from each house. In our familiar geometry, called Euclidean geometry, the set of all points 5 units from a point is commonly known as a circle with radius 5. What is the shape of all points 5 miles from a point in taxicab geometry? This shape can be referred to as a "taxicab circle." If you plot the taxicab circles for 3, 4, 5, 6, 7, and 8 miles from each house, the corresponding intersections will be the points that solve the problem. What is interesting is that in Euclidean geometry the solution is exactly the same: The set of all points equidistant from two points is the set of intersections of all circles of like size.

Finally, students may wonder why the set of points in the solution is not quite in a straight path. Suggest that they try other locations for Clarissa's house, beginning first at $(3, 3)$. Keep Katie's house at the origin, $(0, 0)$, for simplicity and to better observe patterns.

Taxicab geometry is an example of *non-Euclidean* geometry. It is fun to explore ideas in Euclidean geometry and see how they turn out in taxicab geometry. Activity 7.21 was one example. Definitions of line, angle, square, and circle can be applied to taxicab geometry with interesting results. Are there parallel lines and perpendicular lines in taxicab geometry? What is the shortest distance between two points? Is that path unique? *Taxi-Cab Geometry: An Adventure in Non-Euclidean Geometry* by Krause (1987) provides an excellent introduction to non-Euclidean geometry through activities that can be used with elementary and middle school students.

So far, the location activities and tasks in this section have used coordinates to identify locations on a grid and to examine distances between points. In grades 5–8, students can also use coordinates to describe transformations on a coordinate grid. This can be done as an alternative method of introducing slides, flips, and turns or as a follow-up to that discussion.

Coordinates and Transformations

In grades 5–8, shapes and properties can be investigated on a full coordinate grid. Even fifth-grade students can use negative integers so that all four quadrants of the plane can be represented. The activities here suggest how coordinates can be used to examine transformations.

ACTIVITY 7.22

Coordinate Reflections

Have students draw a five-sided shape in the first quadrant on coordinate grid paper using grid points for vertices. Label the figure ABCDE and call it Figure 1. Use the y -axis as a line of symmetry and draw the reflection of the

(continued)

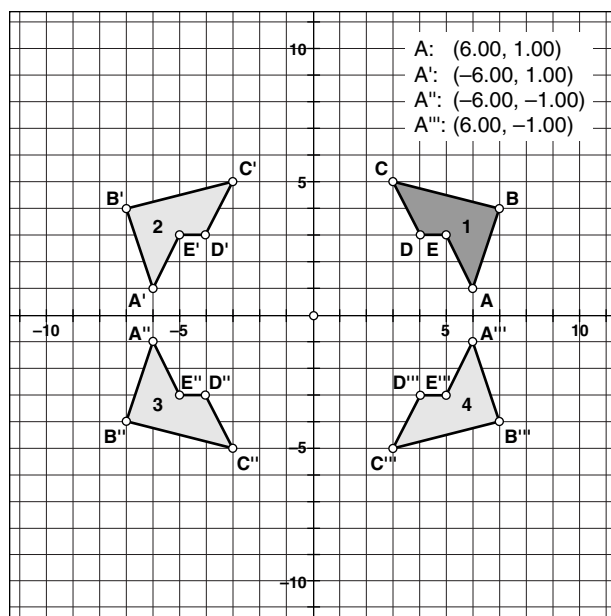


FIGURE 7.32

Figure 1 (ABCDE) is reflected across the y -axis. Then both Figures 1 and 2 are reflected across the x -axis.

shape in the second quadrant. Call it Figure 2 (for second quadrant) and label the reflected points $A'B'C'D'E'$. Now use the x -axis as the line of symmetry. Reflect both Figure 2 and Figure 1 into the third and fourth quadrants, respectively, and call these Figures 3 and 4. Label the points of these figures with double and triple primes (A'' and A''' , and so on). Write in the coordinates for each vertex of all four figures.

- How is Figure 3 related to Figure 4? How else could you have gotten Figure 3? How else could you have found Figure 4?
- How are the coordinates of Figure 1 related to its image in the y -axis, Figure 2? What can you say about the coordinates of Figure 4?
- Make a conjecture about the coordinates of a shape reflected in the y -axis and a different conjecture about the coordinates of a shape reflected in the x -axis.
- Draw lines from the vertices of Figure 1 to the corresponding vertices of Figure 2. What can you say about these lines? How is the y -axis related to each of these lines?

STOP

Use Figure 7.32 or a drawing of your own to answer the questions in Activity 7.22.

Rotations can be explored in a similar manner to Activity 7.22. Have students begin with a figure in the first quadrant, rotate it a quarter turn clockwise and a quarter turn counterclockwise into the second and fourth quadrants, using the origin as the center of rotation. Then have students rotate the original figure a half turn into the third quadrant. Similar questions can be asked about the coordinates. An interesting result can be seen in the figure in the third quadrant.

STOP

There is a relationship between a 180-degree turn and reflections through two perpendicular lines. Can you tell what that is?

In the activities just discussed, students perform a transformation on a figure and observe how the transformation affects the coordinates. In the next two activities, the coordinates will be changed and students observe the resulting change on the figure. Get a coordinate grid and follow along with the next activity.

ACTIVITY 7.23

Coordinate Translations

Students begin with a five-sided shape in the first quadrant. Make a list of the coordinates. Next, students make a new set of coordinates by adding 3 to all of the x -coordinates of the initial figure and then they draw this figure. They then create a third figure by subtracting 4 from each of the y -coordinates of

the original figure. What happens? What will happen if you do both: add 3 to the x -coordinates and subtract 4 from the y -coordinates? Try it. Students now have the original and three image figures on their paper. Now have them draw lines from the vertices of the original figure to the corresponding vertices of the last figure (the one in which both coordinates were changed). What do you notice about these lines? (They are parallel and the same length.) In general, what does adding or subtracting a fixed value to all coordinates do to a figure?

Students who have done the two preceding activities and the variation for rotational symmetry should have a general way to describe translations, reflections across an axis, and 90-degree rotations about the origin all in terms of coordinates. In the following activity, multiplying a constant times the coordinates makes a different type of change.

ACTIVITY 7.24

Coordinate Dilations

Students begin with a four-sided shape in the first quadrant. They then make a list of the coordinates and make a new set of coordinates by multiplying each of the original coordinates by 2. They plot the resulting shape. What is the result? Now have students multiply each of the original coordinates by $\frac{1}{2}$ and plot that shape. What is the result? Next, students draw a line from the origin to a vertex of the largest shape on their paper. Repeat for one or two additional vertices and ask for observations. (An example is shown in Figure 7.33.)



How do the lengths of the sides and the areas of the shapes compare when the coordinates are multiplied by 2? What if they are multiplied by 3 or by $\frac{1}{2}$?

It is impressive to see how an arithmetic operation can control a figure. Imagine being able to control slides, flips, turns, and dilations, not just in the plane but also for three-dimensional figures. The process is identical to and is the essence of computer animation techniques.

When the coordinates of a shape are multiplied as in the preceding activity, each by the same factor, the shape gets either larger or smaller. The size is changed but not the shape. The new shape is similar to the old shape. This is called a *dilation*, a transformation that is *not* rigid because the shape changes.

Your students may enjoy exploring this phenomenon a bit further because they can get some quite interesting effects. If they start with a line drawing of a simple face, boat, or some other shape drawn with straight lines connecting vertices, they see an interesting effect by multiplying just the first coordinates, just the

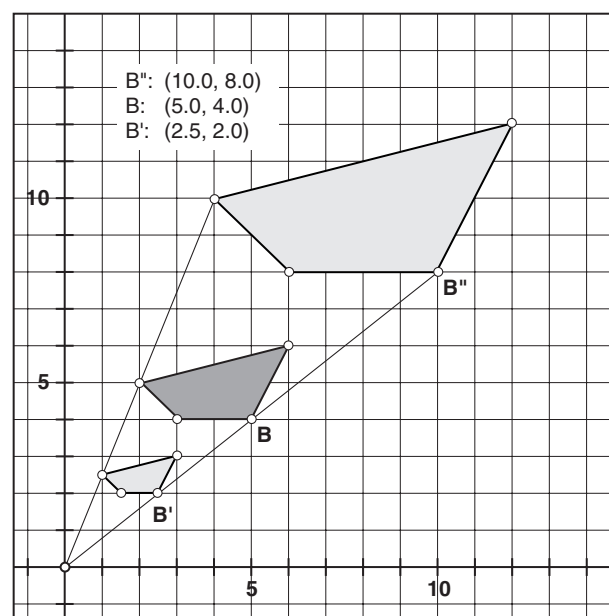
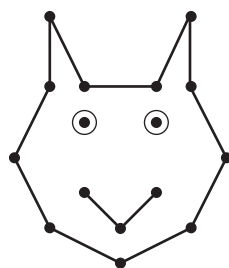


FIGURE 7.33 Dilations with coordinates. Coordinates of the center figure were multiplied by 2 and also by 0.5 to create the other two figures.



second coordinates, or using a different factor for each. For example, students could sketch the cat shown here on a coordinate grid. Then suggest that they add 10 to each first coordinate and multiply each second coordinate by 3.

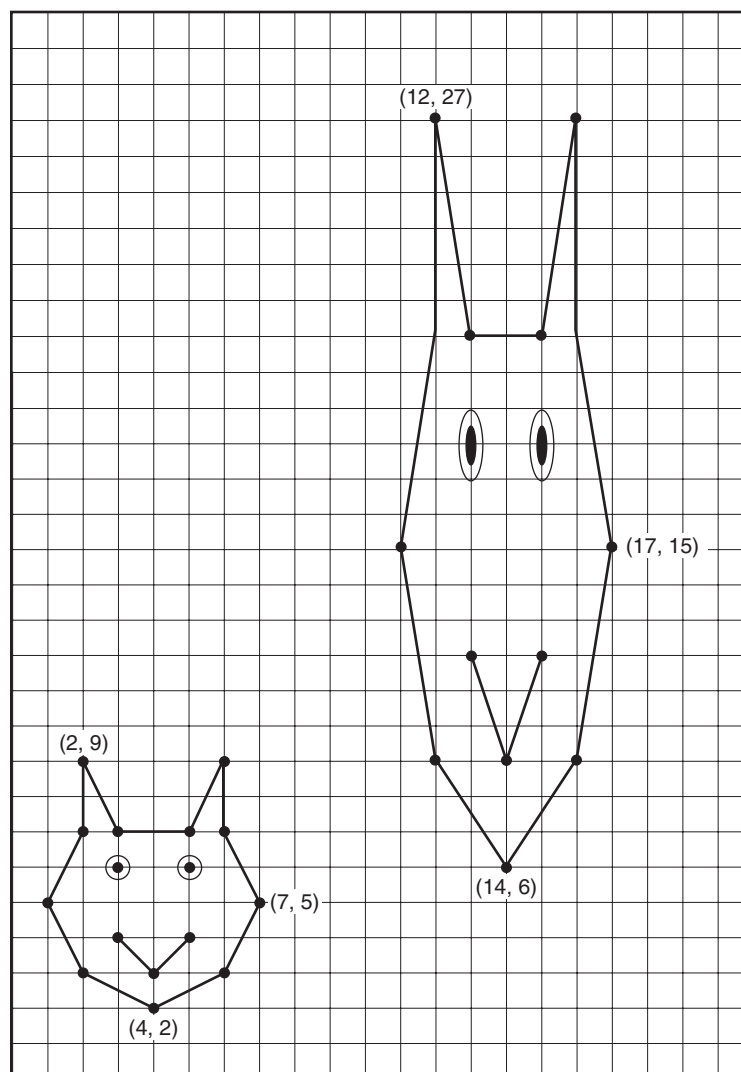


Before reading on, predict what you think will happen to the cat if you add 10 to the first coordinates and multiply the second coordinates by 3. What will happen if you do the reverse—multiply the first coordinates by 3 and add 10 to the second coordinates?

Multiplication of both coordinates by a factor greater than one dilates the figure to a similar but larger shape. When only one coordinate is multiplied, the vertical dimensions alone are dilated, so the figure is proportionately stretched in a vertical manner as in Figure 7.34. By adding 10 units to the first coordinates the new stretched figure is moved to the right so as not to interfere with the original. Students will enjoy exploring with variations of stretching and dilating shapes in various ways.

FIGURE 7.34

Using coordinates to cause a distortion. The coordinates for the larger cat are based on those of the smaller: $(x + 10, 3y)$.



Applying the Pythagorean Relationship

The geometric version of the Pythagorean relationship is about areas. The following activity has students use the coordinate grid and the Pythagorean relationship to develop a formula for the distance between points.

ACTIVITY 7.25

The Distance Formula

Have students draw a line between two points in the first quadrant that are not on the same horizontal or vertical line. The task is to use only the coordinates of the endpoints to calculate the distance between them in terms of the units on the grid. To this end, suggest that they draw a right triangle using the line as the hypotenuse. The vertex at the right angle will share one coordinate from each endpoint. Students can compute the areas of the squares on the legs and add to find the area of the square on the hypotenuse. The length of the original line segment (the distance between the points) is the number whose square is the area of the square on the hypotenuse. (This last sentence is a geometric interpretation of square root.) By computing the areas of these squares students can compute the length of the original line.

Have students look through all of their calculations and see how the coordinates of the two endpoints were used. Challenge students to use the same type of calculations to get the distance between two new points without drawing any pictures.

Although most students in the middle grades are unable to construct proofs, they should be able to follow the rationale if shown proofs. By leading students through the procedure of finding the length of one line (or the distance between the endpoints), you give them sufficient information to compute the lengths of other lines. Students will see that all they need are the coordinates of the two endpoints to compute the areas of all three squares and, hence, the length of the hypotenuse. If you then help them substitute letters for specific coordinates, a general distance formula results.

Slope

The topic of slope is another important connection between geometry and algebra. To begin a discussion of slope, draw several different slanted lines on the board. Discuss how they are different. Some are steeper than others. Some go up, others go down. If you agree that “up” means sloping upward from left to right, then you can agree which ones go up and which go down. This “steepness” of a line is an attribute that can be measured like other measurable attributes. To give slope a number requires a reference line. The coordinate grid provides a reference (the x -axis) and the numbers to use in the measurement. Spend some time having students invent their own methods for attaching a number to the concept of steepness.

The convention for measuring the steepness of a line or the *slope* is based on the ideas of the *rise* and *run* between any two points on the line. The *rise* is the vertical change from the left point to the right point—positive if up, negative if down. The *run* is the horizontal distance from the left point to the right point. Slope is then defined as $\text{rise} \div \text{run}$ or the ratio of the vertical change to the horizontal change (see Figure 7.35). By agreement, vertical lines have no slope or the slope is said to be “undefined.” Horizontal lines have a slope of 0 as a result of the definition.

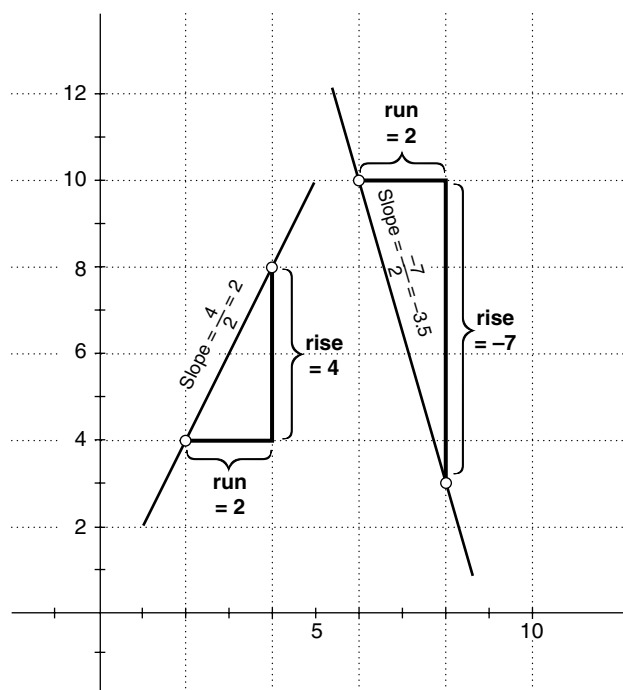


FIGURE 7.35

The slope of a line is $\text{rise} \div \text{run}$.

Once students are given the definition, they should be able to compute the slopes of any nonvertical line drawn on a coordinate grid without further assistance and *without formulas*.

A good problem-based task is to figure out what can be said about the slopes of parallel lines and perpendicular lines. (Is this geometry or algebra? Does it matter?)

Visualization Activities

Visualization activities involve seeing and understanding shapes from different perspectives.

Finding out how many different shapes can be made with a given number of simple tiles demands that students mentally flip and turn shapes in their minds and find ways to decide if they have found them all. Although pentominoes have been around for decades, as a visualization activity they remain superb. If your students are not familiar with pentominoes, the following activity is strongly recommended.

ACTIVITY 7.26

Pentominoes

A pentomino is a shape formed by joining five squares as if cut from a square grid. Each square must have at least one side in common with another. Provide students with five square tiles and a sheet of square grid paper for recording. Challenge them to see how many different pentomino shapes they can find. Shapes that are flips or turns of other shapes are not considered different. Do not tell students how many pentomino shapes there are. Good discussions will come from deciding if some shapes are really different and if all shapes have been found.

Once students have decided that there are just 12 pentominoes (see Figure 7.29, p. 215), the 12 pieces can then be used in a variety of activities. Paste the grids with the students' pentominoes onto tagboard and let them cut out the 12 shapes. These can be used in the next two activities.

It is also fun to explore the number of shapes that can be made from six equilateral triangles or from four 45-degree right triangles (halves of squares). With the right triangles, sides that touch must be the same length. How many of each of these "ominoes" do you think there are? These variations of pentominoes are excellent for a class that has worked previously with pentominoes and still is in need of an early visualization experience.

Lots of activities can be done with pentominoes. For example, try to fit all 12 pieces into a 6×10 or 5×12 rectangle. Also, each of the 12 shapes can be used as a tessellation tile. Another task is to examine each of the 12 pentominoes and decide which

will fold up to make an open box. For those that are “box makers,” which square is the bottom? Once a “box maker” has been identified, challenge students to write the letters $M-A-T-H$ on the four sides so that the box will spell *MATH* around the sides.

As students work at mentally folding a “box maker” pentomino and attempting to correctly orient the letters of *MATH* on the sides, they are making connections between the two-dimensional world and the three-dimensional world. If a “box maker” had one more square attached at an appropriate place, you could fold it up to make a cube. A flat shape that will fold up to make a solid figure is called a *net* of that solid. The following activity suggests several challenges involving nets.

ACTIVITY 7.27

Net Challenges

The following tasks are only related because each involves nets of solids.

- For each of the pentomino “box makers,” see how many different places a sixth square can be attached to create a net for a cube. Are there other nets for a cube that do not begin with a pentomino?
- Begin with a solid, such as a rectangular prism or square pyramid. Sketch as many nets as possible for this shape. Add to the collection some arrangements of the sides of the solid that are not nets. Challenge a friend to decide which are nets of the shape and which are not.
- Use Polydrons or 3D Geoshapes to create a flat figure that you think will fold up into a solid. Test the result. If the number and/or the type of flat shapes is specified, the task can be made more or less difficult. Can you make a net of a solid with 12 regular pentagons or 8 equilateral triangles? (These can be made into a *dodecahedron* and an *octahedron*, respectively, two of the five completely regular polyhedra, also known as the five *Platonic solids*.)

The following activity also provides students with experiences in the three-dimensional world but in a rather different manner. Here students also must mentally move shapes and predict the results. The activity combines ideas of line symmetry and (reflections) as well as visualization and spatial reasoning.

ACTIVITY 7.28

Notches and Holes

Use a half sheet of paper that will easily fit on the overhead. Fold it in half and then half again, making the second fold in the opposite direction from the first. Students make a sketch of the paper when it is opened, showing a line for each fold. With the paper folded, cut notches in one or two sides and/or cut off one or two corners. You can also use a paper punch to make a hole or two. While still folded, place the paper on the overhead showing the notches and holes. The folded edges should be to the left and at the top. (See Figure 7.36.) The task is for students to draw the notches and holes that they think will appear when you open the paper.

To introduce this activity, begin with only one fold and only two cuts. Stay with one fold until students are ready for a more difficult challenge.

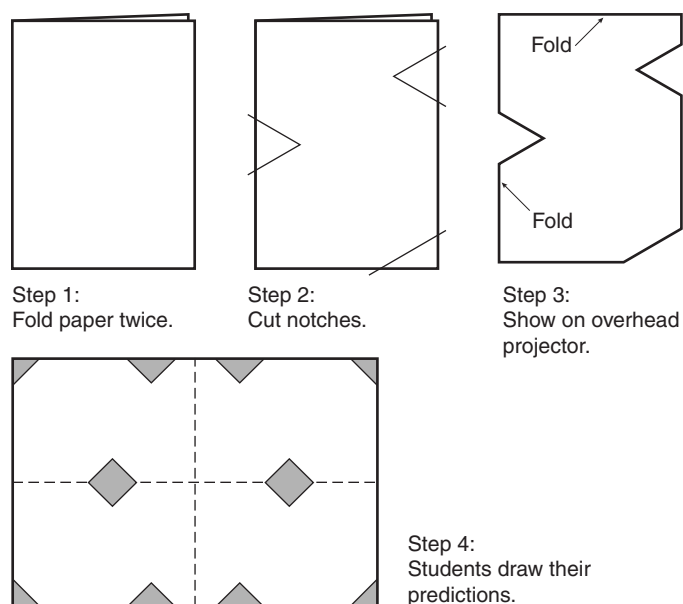


FIGURE 7.36

An example showing how the “Notches and Holes” activity is done. Students make two folds and cut notches and/or punch holes in the folded paper. Before unfolding, they draw a sketch predicting the result when the paper is unfolded.

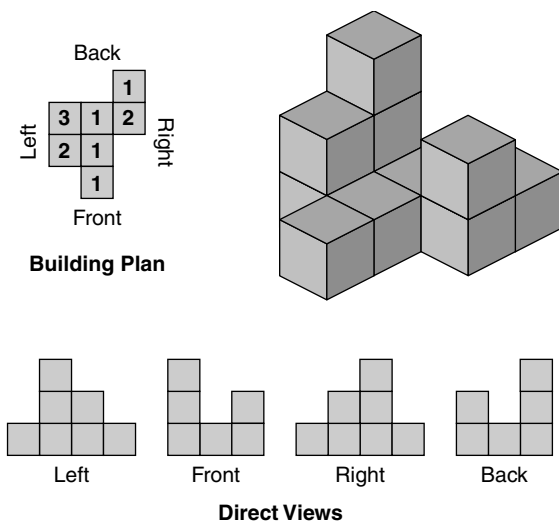


FIGURE 7.37

Tasks can begin with the Building Plan, or with the Direct View, or even with the building. Students give the other representations.

In “Notches and Holes” students will eventually learn which cuts create holes and how many and which cuts make notches in the edges or on the corners. Notice how line symmetry, or reflections, plays a major role in the activity. Symmetry determines the position, the shape, and the number of holes created by each cut.



Stop now and try the “Notches and Holes” activity yourself. Try cuts in various places on the folded paper and see what is involved in this well-known activity.

One of the main goals of the Visualization strand of the Geometry Standard is to be able to identify and draw two-dimensional images of three-dimensional figures and to build three-dimensional figures from two-dimensional images. Activities aimed at this goal often involve drawings of small “buildings” made of 1-inch cubes.

ACTIVITY 7.29

Viewpoints

- In the first version, students begin with a building and draw the left, right, front, and back direct views. In Figure 7.37, the building plan shows a top view of the building and the number of blocks in each position. After students build a building from a plan like this, their task is to draw the front, right, left, and back direct views as shown in the figure.
- In the reverse version of the task, students are given right and front views. The task is to build the building that has those views. To record their solution, they draw a building plan (top view with numbers).

Notice that front and back direct views are symmetric, as are the left and right views. That is why only one of each is given in part b of the activity.

In “Viewpoints,” students made “buildings” out of 1-inch cubes and coordinated these with direct views of the sides and top. A significantly more challenging activity is to draw perspective views of these block buildings or to match perspective drawings with a building. Isometric dot grids are used for the drawings. The next activity provides a glimpse at this form of visualization activity.

ACTIVITY 7.30

Perspective Drawings

- In the first version, students begin with a perspective drawing of a building. The assumption is that there are no hidden blocks. From the drawing the students build the actual building with their blocks. To record the result, they draw a building plan indicating the number of blocks in each position.
- In the second version, students are given either a block plan or the five direct views. They build the building accordingly and draw two or more of the perspective views. There are four possible perspectives from above the table: the front left and right, and the back left and right. It is useful to build the building on a sheet of paper with the words *front*, *back*, *left*, and *right* written on the edges to keep from getting different viewpoints confused.

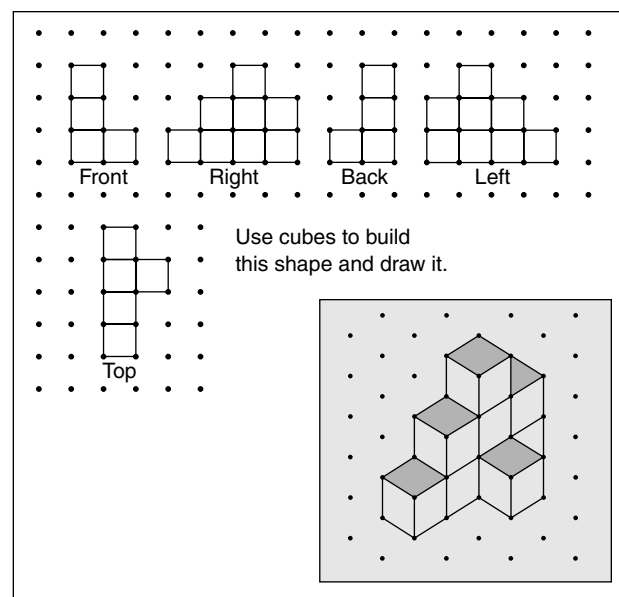


FIGURE 7.38

Develop perspective and visual perception with cubes and plain views. Draw block “buildings” on isometric grids.

Figure 7.38 shows an example of this last activity. Some excellent resource books exist for this type of activity. It is not necessary to prepare these tasks yourself. Perhaps the best-known book is *Middle Grades Mathematics Project: Spatial Visualization* (Winter, Lappan, Phillips, & Fitzgerald, 1986). NCTM’s *Navigating Through Geometry* books have similar activities in both the 3–5 and 6–8 grade books.



Technology Note

An amazing computer tool, the “Isometric Drawing Tool,” for drawing perspective views of block buildings such as in Activity 7.30 is available on the *Illuminations* website (www.illuminations.nctm.org/tools/isometric/isometric.asp). This applet requires only mouse clicks to draw either whole cubes, any single face of a cube, or just lines. The drawings, however, are actually “buildings” and can be viewed as three-dimensional objects. They can be rotated in space so that they can be seen from any vantage. Prepared investigations are informative and also lead students through the features of the tool.

Another interesting connection between two and three dimensions is found in slicing solids in different ways. When a solid is sliced into two parts, a two-dimensional figure is formed on the slice faces. Figure 7.39 shows a cube being sliced off at the corner, leaving a triangular face. Slices can be explored with clay sliced with a potter’s wire as shown in the figure. A niftier method is to partially fill a plastic solid with water. The surface of the water is the same as the face of a slice coinciding with the surface of the water. By tilting the shape in different ways, every possible “slice” can be observed. Small plastic solids such as *Power Solids* are excellent for this.

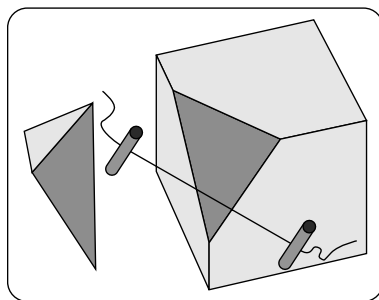
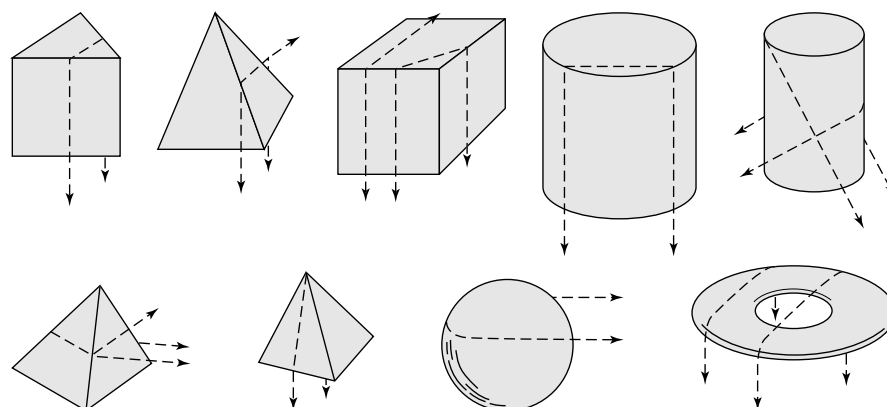


FIGURE 7.39

Predict the slice face before you cut a clay model with a potter's wire.



ACTIVITY 7.31

Water Slices

Students are given a solid and challenged to find out how to slice it to make a designated slice face. The list of challenges should include some that are impossible. Before water is poured into the solids, students must commit themselves on paper as to whether or not the slice is possible. If they think it is possible, they write and/or draw a description of where the slice must be made. Then water is put into the shape to test their predictions. If there are no plastic shapes available, clay models are the next best option. Here is a list of potential slice faces for a cube:

Square, nonsquare rectangle, nonrectangular parallelogram, isosceles trapezoid, nonisosceles trapezoid, equilateral triangle, isosceles right triangle, other isosceles triangle, scalene right triangle, other scalene triangle.

A tetrahedron, a square pyramid, and prisms offer similar challenges. For other solids, it may be more interesting to find out how many different types of slice faces can be found and to describe each.

A *polyhedron* is a three-dimensional shape with polygons for all faces. Among the various polyhedra, the Platonic solids are especially interesting. *Platonic solids* is the name given to the set of completely regular polyhedrons. *Completely regular* means that each face is a regular polygon and every vertex has exactly the same number of faces joining at that point. An interesting visualization task appropriate for this level is to find and describe all of the Platonic solids.

ACTIVITY 7.32

Search for the Platonic Solids

Provide students with a supply of equilateral triangles, squares, regular pentagons, and regular hexagons from one of the plastic sets for building solids (e.g., *Polydron* or *Geofix*). Explain what a completely regular solid is. The task is to find as many different completely regular solids as possible.

One approach to conducting this activity is to leave it as stated and allow students to work with no additional guidance. Success will depend on their problem-solving skills. Alternatively, you might suggest a systematic approach as follows. Since the smallest number of sides a face can have is three, begin with triangles, then squares, then pentagons, and so on. Furthermore, since every vertex must have the same number of faces, try three faces at a point, then four, and so on. (It is clearly impossible to have only two faces at a point.)

With this plan, students will find that for triangles they can have three, four, or five triangles coming to a point. For each of these, they can begin with a “tent” of triangles and then add more triangles so that each vertex has the same number. With three at a point you get a four-sided solid called a *tetrahedron* (*tetra* = four). With four at each point you get an eight-sided solid called an *octahedron* (*octa* = eight). It is really exciting to build the shape with five triangles at each point. It will have 20 sides and is called an *icosahedron* (*icosa* = twenty).

In a similar manner, students will find that there is only one solid made of squares—three at each point and six in all—a *hexahedron* (*hex* = six), also called a cube. And there is only one solid with pentagons, three at each point, 12 in all. This is called a *dodecahedron* (*dodeca* = twelve).



Why are there no regular polyhedra with six or more triangles or four or more squares? Why are there no regular polyhedra made with hexagons or with polygons with more than six sides? The best way to answer these questions is to experiment with the polygons and explain the answers in your own words. Students should do the same.

A fantastic skeletal icosahedron can be built out of the newspaper rods described earlier. (See Figure 7.6, p. 191.) Since five triangles converge at each point, there are also five edges emerging at each point. Simply work at bringing five rods to each vertex and remember that each face is a triangle. This icosahedron will be about 4 feet across and will be amazingly sturdy.



Assessment Note

Principles and Standards is extremely helpful for articulating growth in geometric content across the grades. For each of the four goals articulated by the standards (shape and properties, location, transformations, and visualization) examine the 3–5 grade-level expectations as well as those for the 6–8 grade band. (See Appendix A, p. 352.)

Try to include a sense of growth over time in your assessment of content. If you limit your assessment to a mastery of skills or definitions, the spirit of exploration that you want in your geometry program will be lost. We most often teach in a manner that reflects our assessment plans. Although mastery of some ideas is perhaps important, conceptual development is rarely reflected in memorizing definitions. In deciding what to assess and how, it is best to take a long-term view of geometry rather than a more traditional mastery-oriented approach.

EXPANDED LESSON



Triangle Midsegments

Based on Activity 7.13, p. 208

GRADE LEVEL: Eighth grade.

MATHEMATICS GOALS

- To look at the relationship between a triangle's midsegment and its base.
- To develop the rationale for why particular relationships exist in a triangle.
- To develop logical reasoning in a geometric context.

THINKING ABOUT THE STUDENTS

- Most students are beginning to function at the van Hiele level 2, where they are ready to grapple with "why" and "what-if" questions. Students are aware of the properties of angles formed by cutting parallel lines with a transverse line. They also have experience working with similar triangles.

- To do the lesson with a dynamic geometry program, students should be relatively competent with the program tools and be able independently to draw different geometric objects (e.g., triangles, lines, line segments), label vertices, find midpoints, and measure lengths and angles.

MATERIALS AND PREPARATION

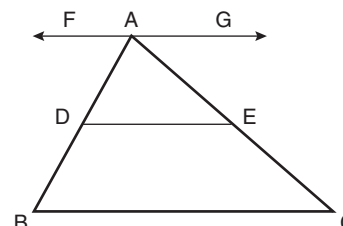
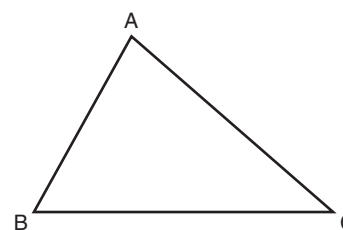
- This lesson can be done either with computers or as a paper-and-pencil task. As described, the lesson only assumes a demonstration computer with display screen. Although desirable, a computer is not required.
- The computer used in the lesson requires a dynamic geometry program, such as *The Geometer's Sketchpad*.
- Students will need rulers that measure in centimeters.

Lesson

BEFORE

Brainstorm

- Have each student draw a line segment measuring 16 cm near the long edge of a blank sheet of paper. Demonstrate using the computer. Label the segment BC.
- Have students randomly select another point somewhere on their papers but at least a few inches above BC. Illustrate on the computer that you want all students to have very different points. Some might be in the upper left, upper right, near the center, and so on. Have them label this point A and then draw segments AB and AC to create triangle ABC. Do the same on the computer.
- Next, students find the midpoints D and E of AB and AC, respectively, and draw the midsegment DE. Do the same on the computer. You can introduce the term *midsegment* if you wish as the line joining the midpoints of two sides of a triangle. Terminology is not important, however.
- Have students measure their midsegments and report what they find. Amazingly, all students should report a measure of 8 cm. On the computer, measure BC and DE. Move point A all over the screen. The two measures will stay the same, with the length of DE being half of BC. Even if B or C is moved, the ratio of BC to DE will remain 2 to 1.
- Ask students for any conjectures they may have about why this relationship exists. Discuss each idea briefly but without any evaluation.
- On the computer draw a line through A parallel to BC. Have students draw a similar line on their paper. Label points F and G on the line as shown here.



- Ask students what else they know about the figure now that line FG has been added. List all ideas on the board.

The Task

- What conjecture can be made about the midsegment of any triangle?
- What reason can be given for why the conjecture might be true?

Establish Expectations

- Students are to write out a conjecture about the midsegment of a triangle.
- In pairs, students are to continue to explore their sketch, looking for reasons why this particular relationship between the midsegment of a triangle and its base exists. They should record all of their ideas and be ready to share them with the class. If they wish to explore an idea on the computer sketch, they should be allowed to do so.

DURING

- Resist giving too much guidance at first. See what students can do on their own. Notice what they focus on in forming conjectures.
- For students having difficulty, suggest that they focus on angles ADE and ABC as well as angles AED and ACB. (These pairs of angles are congruent.)
- Suggest that they list all pairs of angles that they know are congruent. Why are they congruent?
- If necessary, ask students to compare triangle ABC with triangle ADE. What do they notice? They should note that the triangles are similar. Why are they similar?

AFTER

- Have students discuss their initial arguments for why the midsegment relationship holds. They can use the demonstration computer to share their ideas.
- Using their ideas, help students build arguments so that one can see how one point flows to the next in a logical sequence.

ASSESSMENT NOTES

- Look for students who struggle seeing the connections or relationships between properties. They may not be functioning at level 2 of the van Hiele levels of geometric thought.
- Do students see the difference between simply observing a relationship and considering the reasons the relationship exists?

- If students are not able to follow logical arguments (yours or their peers') in this activity, they may not yet be at the van Hiele level 2. These students need more informal experiences exploring properties of shapes and looking for relationships. Select from the level 1 activities for Shapes and Properties beginning on p. 193.
- If this activity is successful, continue having students grapple with making logical arguments for why

the things they observe to be true are in fact true. Activity 7.12, "Angle Sum in a Triangle," provides students another opportunity to consider the reasoning behind a geometric relationship. The arguments there are similar to those in this activity.

next steps