Chapter 6

STRATEGIES FOR WHOLE-NUMBER COMPUTATION

Much of the public sees computational skill as the hallmark of what it means to know mathematics at the elementary school level. Although this is far from the truth, the issue of computational skills with whole numbers is, in fact, a very important part of the elementary curriculum, especially in grades 2 to 6.

Rather than constant reliance on a single method of subtracting (or any operation), methods can and should change flexibly as the numbers and the context change. In the spirit of the *Standards,* the issue is no longer a matter of "knows how to subtract three-digit numbers"; rather it is the development over time of an assortment of flexible skills that will best serve students in the real world.

big ideas

1. **Flexible methods of computation involve taking apart and combining numbers in a wide variety of ways. Most of the partitions of numbers are based on place value or "compatible" numbers—number pairs that work easily together, such as 25 and 75.**
2. **Invented strategies are flexible methods of computing that vary with the numbers and the situation. Successful use of the strategies requires that they be understood by the one who is using them—hence, the term *invented.* Strategies may be invented by a peer or the class as a whole; they may even be suggested by the teacher. However, they must be constructed by the student.**
3. **Flexible methods for computation require a good understanding of the operations and properties of the operations, especially the turnaround property and the distributive property for multiplication. How the operations are related—addition to subtraction, addition to multiplication, and multiplication to division—is also an important ingredient.**
4. **The traditional algorithms are clever strategies for computing that have been developed over time. Each is based on performing the operation on one place value at a time with transitions to an adjacent position (trades, regrouping, "borrows," or "carries"). These algorithms work for all numbers but are often far from the most efficient or useful methods of computing.**

Toward Computational Fluency

With today's technology the need for doing tedious computations by hand has essentially disappeared. At the same time, we now know that there are numerous methods of computing that can be handled either mentally or with pencil-and-paper support. In most everyday instances, these alternative strategies for computing are easier and faster, can often be done mentally, and contribute to our overall number sense. The traditional algorithms (procedures for computing) do not have these benefits.

Consider the following problem.

Mary has 114 spaces in her photo album. So far she has 89 photos in the album. How many more photos can she put in before the album is full?

STOP **Try solving the photo album problem using some method other than the one you were taught in school. If you want to begin with the 9 and the 4, try a different approach. Can you do it mentally? Can you do it in more than one way? Work on this before reading further.**

Here are just four of many methods that have been used by students in the primary grades to solve the computation in the photo album problem:

89 + 11 is 100. 11 + 14 is 25.

90 + 10 is 100 and 14 more is 24 plus 1 (for 89, not 90) is 25.

Take away 14 and then take away 11 more or 25 in all.

89, 99, 109 (that's 20). 110, 111, 112, 113, 114 (keeping track on fingers) is 25.

Strategies such as these can be done mentally, are generally faster than the traditional algorithms, and make sense to the person using them. Every day, students and adults resort to error-prone, traditional strategies when other, more meaningful methods would be faster and less susceptible to error. Flexibility with a variety of computational strategies is an important tool for successful daily living. It is time to broaden our perspective of what it means to compute.

Figure 6.1 lists three general types of computing. The initial, inefficient direct modeling methods can, with guidance, develop into an assortment of invented strategies that are flexible and useful. As noted in the diagram, many of these methods can be handled mentally, although no special methods are designed specifically for mental computation. The traditional pencil-and-paper algorithms remain in the mainstream curricula. However, the attention given to them should, at the very least, be debated.

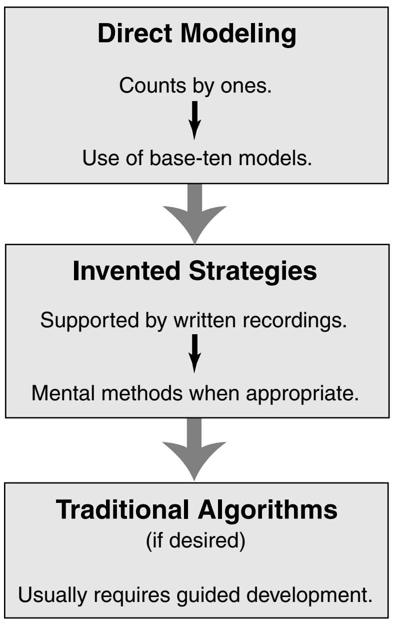

FIGURE 6.1

Three types of computational strategies.

Direct Modeling

The developmental step that usually precedes invented strategies is called *direct modeling:* the use of manipulatives or drawings along with counting to represent directly the meaning of an operation or story problem. Figure 6.2 provides an example using base-ten materials, but often students use simple counters and count by ones.

Students who consistently count by ones most likely have not developed base-ten grouping concepts. That does not mean that they should not continue to solve problems involving two-digit numbers. As you work with these children, suggest (don't force) that they group counters by tens as they count. Perhaps instead of making large piles, they might make bars of ten from connecting cubes or organize counters in cups of ten. Some students will use the ten-stick as a counting device to keep track of counts of ten, even though they are counting each segment of the stick by ones.

When children have plenty of experience with base-ten concepts and models, they begin to use these ideas in the direct modeling of the problems. Even when students use base-ten materials, they will find many different ways to solve problems.

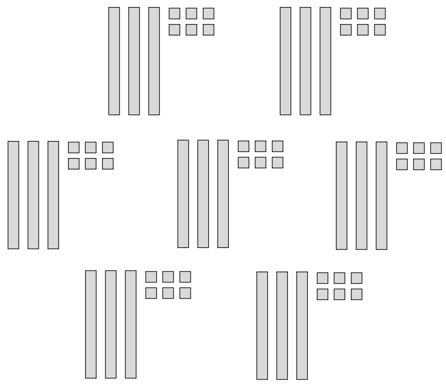

FIGURE 6.2

A possible direct modeling of 36 × 7 using base-ten models.

Invented Strategies

We will refer to any strategy other than the traditional algorithm and that does not involve the use of physical materials or counting by ones as an *invented strategy.* These invented strategies might also be called *personal and flexible strategies*. At times, invented strategies are done mentally. For example, 75 + 19 can be done mentally (75 + 20 is 95, less 1 is 94). For 847 + 256, some students may write down intermediate steps to aid in memory as they work through the problem. (Try that one yourself.) In the classroom, some written support is often encouraged as strategies develop. Written records of thinking are more easily shared and help students focus on the ideas. The distinction between written, partially written, and mental is not important, especially in the development period.

Over the past two decades, a number of research projects have focused attention on how children handle computational situations when they have not been taught a specific algorithm or strategy. Three elementary curricula each base the development of computational methods on student-invented strategies. These are often referred to as "reform curricula" (*Investigations in Number, Data, and Space, Trailblazers,* and *Everyday Mathematics*)."There is mounting evidence that children both in and out of school can construct methods for adding and subtracting multidigit numbers without explicit instruction" (Carpenter et al., 1998, p. 4).

Not all students invent their own strategies. Strategies invented by class members are shared, explored, and tried out by others. However, no student should be permitted to use any strategy without understanding it.

Contrasts with Traditional Algorithms

There are significant differences between invented strategies and the traditional algorithms.

1. *Invented strategies are number oriented rather than digit oriented.* For example, an invented strategy for 618 – 254 might begin with 600 – 200 is 400. Another approach might begin with 254. Adding 46 is 300 and then 300 more to 600. In either case, the computation begins with complete three-digit numbers rather than the individual digits 8 – 4 as in the traditional algorithm. Using the traditional algorithm for 45 + 32, children never think of 40 and 30 but rather 4 + 3. Kamii, long a crusader against standard algorithms, claims that they "unteach" place value (Kamii & Dominick, 1998).

2. *Invented strategies are left-handed rather than right-handed.* Invented strategies begin with the largest parts of numbers, those represented by the leftmost digits. For 86 – 17, an invented strategy might begin with 80 – 10, 80 – 20, or perhaps 86 – 10. These and similar left-handed beginnings provide a quick sense of the size of the answer. With the traditional approach, after borrowing from the 8 and computing 16 – 7, all we know is that the answer ends in 9. By beginning on the right with a digit orientation, traditional methods hide the result until the end. Long division is an exception.
3. *Invented strategies are flexible rather than rigid.* As in 1 and 2 above, several different strategies can be used to begin an addition or subtraction computation. Invented strategies also tend to change or adapt to the numbers involved. Try each of these mentally: 465 + 230 and 526 + 98. Did you use the same method? The traditional algorithm suggests using the same tool on all problems. The traditional algorithm for 7000 – 25 typically leads to student errors, yet a mental strategy is relatively simple.

Benefits of Invented Strategies

The development of invented strategies delivers more than computational facility. Both the development of these strategies and their regular use have positive benefits that are difficult to ignore.

- *Base-ten concepts are enhanced.* There is a definite interaction between the development of base-ten concepts and the process of inventing computational strategies (Carpenter et al., 1998). "Invented strategies demonstrate a hallmark characteristic of understanding" (p. 16). The development of invented strategies should be integrated with the development of base-ten concepts, even as early as first grade.
- *Students make fewer errors.* Research has found that when students use their own strategies for computation they tend to make fewer errors because they understand their own methods (e.g., see Kamii and Dominick, 1997). Decades of trying to teach the traditional algorithms, no matter how conceptually, have continually demonstrated that students make numerous, often systematic errors that they use again and again. Systematic errors are not typical with invented strategies.
- *Less reteaching is required.* Students rarely use an invented strategy they do not understand. The supporting ideas are firmly networked with a sense of number, thus making the strategies more permanent. In contrast, students are frequently seen using traditional algorithms without being able to explain why they work (Carroll & Porter, 1997).
- *Invented strategies provide the basis for mental computation and estimation.* Since traditional algorithms are poorly suited to mental computation and estimation strategies, students are forced to temporarily abandon the very strategies that were taught and learn new, number-oriented, left-handed methods. It makes much more sense to teach these methods from the beginning. As students become more and more proficient with these flexible methods using pencil-and-paper support, they soon are able to use them mentally or adapt them to estimation methods. Again we find that time is saved in the curriculum.

- *Flexible, invented strategies are often faster than the traditional algorithms.* It is sad to see a student (or even adults) tediously regrouping for computations such as 300 – 98 or 4 × 75. Much of the computation that adults do daily when technology is not available is of the type that lends itself to methods than can often be done very quickly with a nontraditional approach.
- *Invented strategies serve students at least as well on standard tests.* Evidence suggests that students not taught traditional algorithms fare about as well in computation on standardized tests as students in traditional programs (Campbell, 1996; Carroll, 1996, 1997; Chambers, 1996). As an added bonus, students tend to do quite well with word problems, since they are the principal vehicle for developing invented strategies. The pressures of external testing do not dictate a focus on the traditional algorithms.

Mental Computation

A mental computation strategy is simply any invented strategy that is done mentally. What may be a mental strategy for one student may require written support by another. Initially, students should not be asked to do computations mentally, as this may threaten those who have not yet developed a reasonable invented strategy or who are still at the direct modeling stage. At the same time, you may be quite amazed at the ability of students (and at your own ability) to do computations mentally.

Try your own hand with this example:

342 + 153 + 481

STOP **For the addition task just shown, try this method: Begin by adding the hundreds, saying the totals as you go—*3 hundred, 4 hundred, 8 hundred.* Then add on to this the tens in successive manner and finally the ones. Do it now.**

As your students become more adept, they can and should be challenged from time to time to do appropriate computations mentally. Do not expect the same skills of all students.

Traditional Algorithms

Teachers often ask, "How long should I wait until I show them the 'regular' way?" The question is based on a fear that without learning the same methods that all of us grew up with, students will somehow be disadvantaged. For addition and subtraction this is simply not the case. The primary goal for all computation should be students' ability to compute in some efficient manner—not what algorithms are used. That is, the *method* of computing is not the objective; the ability to compute is the goal. For multiplication and division, many teachers will see a greater need for traditional approaches, especially with three or more digits involved. However, even with those operations, the traditional algorithms are not necessary.

Abandon or Delay Traditional Algorithms

Flexible left-handed methods done mentally with written support are absolutely all that are necessary for addition and subtraction. Developed with adequate practice in the primary grades, these flexible approaches will become mental and very efficient for most students by fifth grade and will serve them more than adequately throughout life. You may find this difficult to accept for two reasons: first, because the traditional algorithms have been a significant part of your mathematical experiences, and second, because *you* may not have learned these skills. These are not reasons to teach the traditional algorithms for addition and subtraction.

For multiplication and division, the argument may not be quite as strong as the number of digits involved increases. However, through the third grade where students need only multiply or divide by a single digit, invented strategies are not only adequate but will provide the benefits of understanding and flexibility mentioned earlier. It is worth noting again that there is evidence that students do quite well on the computation portions of standardized tests even if they are never taught the traditional methods.

If, for whatever reason you feel you must teach the traditional algorithms, consider the following:

- Students will not invent the traditional methods because right-handed methods are simply not natural. This means that you will have to introduce and explain each algorithm.
- No matter how carefully you suggest that these right-handed borrow-and-carry methods are simply another alternative, students will sense that these are the "right ways" or the "real ways" to compute. *This is how Mom and Dad do it. This is what the teacher taught us*. As a result, most students will abandon any flexible left-handed methods they may have been developing.

It is not that the traditional algorithms cannot be taught with a strong conceptual basis. Textbooks have been doing an excellent job of explaining these methods for years. The problem is that the traditional algorithms, especially for addition and subtraction, are not natural methods for students. As a result, the explanations generally fall on deaf ears. Far too many students learn them as meaningless procedures, develop error patterns, and require an excessive amount of reteaching or remediation. If you are going to teach the traditional algorithms, you are well advised to spend a significant amount of time—months, not weeks—with invented methods. Delay! The understanding that children gain from working with invented strategies will make it much easier for you to teach the traditional methods.

Traditional Algorithms Will Happen

You probably cannot keep the traditional algorithms out of your classroom. Children pick them up from older siblings, last year's teacher, or well-meaning parents. Traditional algorithms are in no way evil, and so to forbid their use is somewhat arbitrary. However, students who latch on to a traditional method often resist the invention of more flexible strategies. What do you do then?

First and foremost, apply the same rule to traditional algorithms as to all strategies: *If you use it, you must understand why it works and be able to explain it.* In an atmosphere that says, "Let's figure out why this works," students can profit from making sense of these algorithms just like any other. But the responsibility should be theirs, not yours.

Accept a traditional algorithm (once it is understood) as one more strategy to put in the class "tool box" of methods. But reinforce the idea that like the other strategies, it may be more useful in some instances than in others. Pose problems where a mental

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strategy is much more useful, such as 504 – 498 or 25 + 62. Discuss which method seemed best. Point out that for a problem such as 4568 + 12,813, the traditional algorithm has some advantages. But in the real world, most people do those computations on a calculator.

Development of Invented Strategies: A General Approach

Students do not spontaneously invent wonderful computational methods while the teacher sits back and watches. Among different reform or progressive programs, students tended to develop or gravitate toward different strategies suggesting that teachers and the programs do have an effect on what methods students develop. This section discusses general pedagogical methods for helping children develop invented strategies.

Use Story Problems Frequently

When computational tasks are embedded in simple contexts, students seem to be more engaged than they are with bare computations. Furthermore, the choice of story problems influences the strategies students use to solve them. Consider these problems:

Max had already saved 68 cents when Mom gave him some money for running an errand. Now Max has 93 cents. How much did Max earn for his errand?

George took 93 cents to the store. He spent 68 cents. How much does he have left?

The computation 93 – 68 solves both problems, but the first is more likely than the second to be solved by an add-on method. In a similar manner, fair-share division problems are more likely to encourage a share strategy than a measurement or repeated subtraction problem.

Not every task need be a story problem. Especially when students are engaged in figuring out a new strategy, bare arithmetic problems are quite adequate.

Use the Three-Part Lesson Format

The three-part lesson format described in Chapter 1 is a good structure for an invented-strategy lesson. The task can be one or two story problems or even a bare computation but always with the expectation that the method of solution will be discussed.

Allow plenty of time to solve a problem. Listen to the different strategies students are using, but do not interject your own. Challenge able students to find a second method, solve a problem without models, or improve on a written explanation. Allow children who are not ready for thinking with tens to use simple counting methods. Students who finish quickly may share their methods with others before sharing with the class.

The most important portion of the lesson comes when students explain their solution methods. Help students write their explanations on the board or overhead. Encourage students to ask questions of their classmates. Occasionally have the class try a particular method with different numbers to see how it works.

Remember, not every student will invent strategies. However, students can and will try strategies that they have seen and that make sense to them.

Select Numbers with Care

With the traditional algorithms you are used to distinguishing between problems that require regrouping and those that do not. When encouraging students to develop their own methods, there are more factors to consider. For addition, 35 + 42 is generally easier than 35 + 47. However, 30 + 20 is easier than both and can help students begin to think in terms of tens. Paired with this might be 46 + 10 or 20 + 63. At grade 1, 10 + 11 can provide challenge, a variety of methods, and the start of thinking with tens. Two-digit plus one-digit sums can also serve as a useful stepping-stone. For addition, *Will the sum go over 100?* is a thought to consider.

Think about how multiples of ten might help. For subtraction, learning to add up to a multiple of 10 and especially to 100 is particularly useful. Therefore, tasks such as 30 – 12 and 100 – 35 can provide important readiness for later problems. Tasks such as 417 – 103 or 417 – 98 may each encourage students to subtract 100 and then adjust.

There is no best sequence of problem types. Listen to the strategies students use and select numbers that can build on those ideas or help others in the class to see a new way of thinking. Similar care can and should be given to the selection of multiplication and division tasks.

Integrate Computation with Place-Value Development

In Chapter 5 we made the point that students can begin to develop computational strategies as they are learning about tens and ones. It is not necessary to wait until students have learned place value before they begin computing. Notice how the examples in the preceding section on number selection can help reinforce the way that our number system is built on a structure of groups of tens. In Chapter 5 there is a section entitled "Activities for Flexible Thinking" (p. 145). The activities in that section are appropriate for grades 2 and 3 and complement the development of invented strategies, especially for addition and subtraction.

Progression from Direct Modeling

Direct modeling involving tens and ones can and will lead eventually to invented strategies. However, students may need to be encouraged to move away from the direct modeling process. Here are some ideas:

- Record students' verbal explanations on the board in ways that they and others can model. Have the class follow the recorded method using different numbers.

- Ask children to make a written numeric record of what they did when they solved the problem with models. Explain that they are then going to try to use the same method on a new problem.
- Ask students who have just solved a problem with models to see if they can do it in their heads.
- Pose a problem to the class, and ask students to solve it mentally if they are able.

Invented Strategies for Addition and Subtraction

Research has demonstrated that children will invent a lot of different strategies for addition and subtraction. Your goal might be that each of your children has at least one or two methods that are reasonably efficient, mathematically correct, and useful with lots of different numbers. Expect different children to settle on different strategies.

It is not at all unreasonable for students to be able to add and subtract two-digit numbers mentally by third grade. However, daily recording of strategies on the board not only helps communicate ideas but also helps children who need the short-term memory assistance of recording intermediate steps.

Adding and Subtracting Single Digits

Children can easily extend addition and subtraction facts to higher decades.

Tommy was on page 47 of his book. Then he read 8 more pages. How many pages did Tommy read in all?

If students are simply counting on by ones, the following activity may be useful. It is an extension of the make-ten strategy for addition facts.

ACTIVITY 6.1

Ten-Frame Adding and Subtracting

Quickly review the make-ten idea from addition facts using two ten-frames. (Add on to get up to ten and then add the rest.) Challenge children to use the same idea to add on to a two-digit number as shown in Figure 6.3. Two students can work together. First, they make a specified two-digit number with the little ten-frame cards. They then stack up all of the less-than-ten cards and turn them over one at a time. Together they talk about how to get the total quickly.

The same approach is used for subtraction. For instance, for 53 – 7, take off 3 to get to 50, then 4 more is 46.

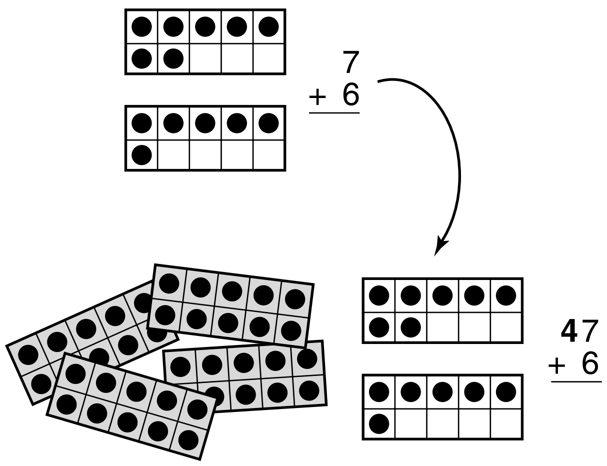
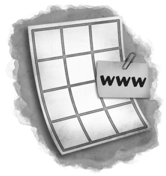


FIGURE 6.3

Little ten-frame cards can help children extend the make-ten idea to larger numbers.



BLMs 17–18

Notice how building up through ten (as in 47 + 6) or down through ten (as in 53 – 7) is different from carrying and borrowing. No ones are exchanged for a ten or tens for ones. The ten-frame cards encourage students to work with multiples of ten without regrouping.

Another important model to use in the second and third grades is the hundreds chart. The hundreds chart has the same tens structure as the little ten-frame cards. For 47 + 6 you count 3 to get out to 50 at the end of the row and then 3 more in the next row.

Adding and Subtracting Tens and Hundreds

Sums and differences involving multiples of 10 or 100 are easily computed mentally. Write a problem such as the following on the board:

$$300 + 500 + 20$$

Challenge children to solve it mentally. Ask students to share how they did it. Look for use of place-value words: "3 *hundred* and 5 *hundred* is 8 *hundred,* and 20 is 820."

Adding Two-Digit Numbers

For each of the examples that follow, a possible recording method is offered. These are intended to be suggestions, not prescriptions. Children have difficulty inventing recording techniques. If you record their ideas on the board as they explain their ideas, you are helping them develop written techniques. You may even discuss recording methods with individuals or with the class to decide on a form that seems to work well. Horizontal formats encourage students to think in terms of numbers instead of digits. A horizontal format is also less likely to encourage use of the traditional algorithms.

Students will often use a counting-by-tens-and-ones technique for some of these methods. That is, instead of "46 + 30 is 76," they may count "46 ⟶ 56, 66, 76." These counts can be written down as they are said to help students keep track.

Figure 6.4 illustrates four different strategies for addition of two two-digit numbers. The following story problem is a suggestion.

FIGURE 6.4

Four different invented strategies for adding two two-digit numbers.

Invented Strategies for Addition with Two-Digit Numbers

Add Tens, Add Ones, Then Combine	**Move Some to Make Tens**
46 + 38 40 and 30 is 70. 6 and 8 is 14. 70 and 14 is 84. 46 +38 70 14 84	46 + 38 Take 2 from the 46 and put it with the 38 to make 40. Now you have 44 and 40 more is 84. 2 ~~46~~ + ~~38~~ 44 + 40 84
Add On Tens, Then Add Ones	**Use a Nice Number and Compensate**
46 + 38 46 and 30 more is 76. Then I added on the other 8. 76 and 4 is 80 and 4 is 84. 46 + 38 → 76 + 8 → 80, 84	46 + 38 46 and 40 is 86. That's 2 extra, so it's 84. 46 + 38 46 + 40 → 86 – 2 → 84

The two Scout troops went on a field trip. There were 46 Girl Scouts and 38 Boy Scouts. How many Scouts went on the trip?

The *move to make ten* and *compensation* strategies are useful when one of the numbers ends in 8 or 9. To promote that strategy, present problems with addends like 39 or 58. Note that it is only necessary to adjust one of the two numbers.

STOP **Try adding 367 + 155 in as many different ways as you can. How many of your ways are like those in Figure 6.4?**

Subtracting by Counting Up

This is an amazingly powerful way to subtract. Students working on the *think-addition* strategy for their basic facts can also be solving problems with larger numbers. The concept is the same. It is important to use *join with change unknown* problems or *missing-part* problems to encourage the counting-up strategy. Here is an example of each.

Sam had 46 baseball cards. He went to a card show and got some more cards for his collection. Now he has 73 cards. How many cards did Sam buy at the card show?

Juanita counted all of her crayons. Some were broken and some not. She had 73 crayons in all. 46 crayons were not broken. How many were broken?

The numbers in these problems are used in the strategies illustrated in Figure 6.5.

Invented Strategies for Subtraction by Counting Up

Add Tens to Get Close, Then Ones

73 – 46

46 and 20 is 66.
(30 more is too much.)
Then 4 more is 70 and 3 is 73.
That's 20 and 7 or 27.

46 > 20
66 > 4
70 > 3
73 — 27

Add Tens to Overshoot, Then Come Back

73 – 46

46 and 30 is 76.
That's 3 too much, so it's 27.

73 – 46
46 + 30 → 76 – 3 → 73
30 – 3 = 27

Add Ones to Make a Ten, Then Tens and Ones

73 – 46

46 and 4 is 50.
50 and 20 is 70 and 3 more is 73. The 4 and 3 is 7 and 20 is 27.

73 – 46
46 + 4 → 50
+ 20 → 70
+ 3 → 73
27

Similarly,

46 and 4 is 50.
50 and 23 is 73.
23 and 4 is 27.

46 + 4 → 50
50 + 23 → 73
23 + 4 = 27

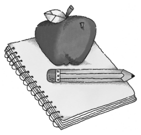
FIGURE 6.5
Subtraction by counting up is a powerful method.

Emphasize the value of using tens by posing problems involving multiples of 10. In 50 – 17, the use of ten can happen by adding up from 17 to 20, or by adding 30 to 17. Some students may reason that it must be 30-something because 30 and 17 is less than 50 and 40, and 17 is more than 50. Because it takes 3 to go with 7 to make 10, the answer must be 33. Work on naming the missing part of 50 or 100 is also valuable. (See Activity 5.23, "The Other Part of 100," p. 147.)

Take-Away Subtraction

Using take-away is considerably more difficult to do mentally. However, take-away strategies are common, probably because traditional textbooks emphasize take-away as the meaning of subtraction and take-away is the basis of the traditional algorithm. Four different strategies are shown in Figure 6.6.

There were 73 children on the playground. The 46 second-grade students came in first. How many children were still outside?



EXPANDED LESSON

(pages 184–185)

The Expanded Lesson for this chapter has students explore subtraction of two-digit numbers.

The two methods that begin by taking tens from tens are reflective of what most students do with base-ten pieces. The other two methods leave one of the numbers intact and subtract from it. Try 83 – 29 in your head by first taking away 30 and adding 1 back. This is a good mental method when subtracting a number that is close to a multiple of ten.

STOP **Try computing 82 – 57. Use both take-away and counting-up methods. Can you use all of the strategies in Figures 6.5 and 6.6 without looking?**

FIGURE 6.6

Take-away strategies work reasonably well for two-digit problems. They are a bit more difficult with three digits.

Invented Strategies for Take-Away Subtraction

Take Tens from the Tens, Then Subtract Ones

73 – 46

70 minus 40 is 30.
Take away 6 more is 24.
Now add in the 3 ones → 27.

73 – 46
70 – 40 → 30 – 6 →
24 + 3 → 27

Or

70 minus 40 is 30.
I can take those 3 away, but I need 3 more from the 30 to make 27.

7~~3~~
– 46
30
– 3
27

Take Away Tens, Then Ones

73 – 46

73 minus 40 is 33.
Then take away 6:
3 makes 30 and
3 more is 27.

73 – 40 → 33 – 3
30 – 3 → 27

Take Extra Tens, Then Add Back

73 – 46

73 take away 50 is 23.
That's 4 too many.
23 and 4 is 27.

73 – 50 → 23 + 4
27

Add to the Whole If Necessary

73 – 46

Give 3 to 73 to make 76.
76 take away 46 is 30.
Now give 3 back → 27.

+3
73 – 46
76 – 46 → 30
– 3 → 27

Extensions and Challenges

Each of the examples in the preceding sections involved sums less than 100 and all involved *bridging a ten;* that is, if done with a traditional algorithm, they require carrying or borrowing. Bridging, the size of the numbers, and the potential for doing problems mentally are all issues to consider.

Bridging

For most of the strategies, it is easier to add or subtract when bridging is not required. Try each strategy with 34 + 52 or 68 – 24 to see how it works. Easier problems instill confidence. They also permit you to challenge your students with a "harder one." There is also the issue of bridging 100 or 1000. Try 58 + 67 with different strategies. Bridging across 100 is also an issue for subtraction. Problems such as 128 – 50 or 128 – 45 are more difficult than ones that do not bridge 100.

Larger Numbers

Most curricula will expect third graders to add and subtract three-digit numbers. Your state standards may even require work with four-digit numbers. Try seeing how *you* would do these without using the traditional algorithms: 487 + 235 and 623 – 247. For subtraction, a counting-up strategy is usually the easiest. Occasionally, other strategies appear with larger numbers. For example, "chunking off" multiples of 50 or 25 is often a useful method. For 462 + 257, pull out 450 and 250 to make 700. That leaves 12 and 7 more ⟶ 719.

Traditional Algorithms for Addition and Subtraction

The traditional computational methods for addition and subtraction are significantly different from nearly every invented method. In addition to starting with the rightmost digits and being digit oriented (as already noted), the traditional approaches involve the concept generally referred to as *regrouping* (a very strange term), exchanging 10 in one place-value position for 1 in the position to the left ("carrying")—or the reverse, exchanging 1 for 10 in the position to the right ("borrowing"). The terms *borrowing* and *carrying* are obsolete and conceptually misleading. The word *regroup* also offers no conceptual help to young children. A preferable term is *trade*. Ten ones are *traded* for a ten. A hundred is *traded* for 10 tens.

Terminology aside, the trading process is quite different from the bridging process used in all invented and mental strategies. Consider the task of adding 28 + 65. Using the traditional method, we first add 8 and 5. The resulting 13 ones must be separated into 3 ones and 1 ten. The newly formed ten must then be combined with the other tens. This process of "carrying a ten" is conceptually difficult and is different from the bridging process that occurs in invented strategies. In fact, nearly all major textbooks now teach this process of regrouping prior to and separate from direct instruction with the addition and subtraction algorithm, an indication of the difficulties involved. The process is even more difficult for subtraction, especially across a zero in the tens place where two successive trades are required.

Compounding all of this is the issue of recording each step. The traditional algorithms do not lend themselves to mental computation and so students must learn to record. The literature of the past 50 years is replete with the errors that students make with these recording methods.

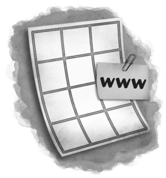
It is also a serious error to focus on nonregrouping problems before tackling regrouping. Teaching nonregrouping problems first causes bad habits that children must later unlearn. The most common result is for students to completely ignore the numbers involved and focus only on adding in each column. When regrouping is eventually introduced, there is a strong tendency to not regroup at all, recording 615 as the sum of 28 and 47. When carrying is finally emphasized, many students compensate by carrying all of the time, even when not needed. Similar difficulties arise in subtraction.

All of these observations are offered to encourage you to abandon the traditional algorithms for addition and subtraction and, failing that, to alert you to the difficulties that your students will likely experience. Having said that, we offer some guidance for you if you must teach the standard procedures. Since it will never occur to students to add or subtract beginning in the ones place, you will have to use a more direct approach to instruction rather than a strictly problem-oriented approach.

The Addition Algorithm

Explain to the students that they are going to learn a method of adding that most "big people" learned when they were in school. It is not the only way or even the best way; it is just a method you want them to learn.

Begin with Models Only



BLM 15

In the beginning, avoid any written work except for the possible recording of an answer. Provide children with place-value mats and base-ten models. A mat with two ten-frames in the ones place (Blackline Masters) is suggested.

Have students make one number at the top of the mat and a second beneath it as shown in Figure 6.7. A groupable model such as counters in cups is most helpful.

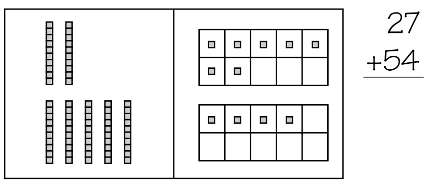
You will need to explain two rules. First: *Begin in the ones column.* Second: *Do not keep more than 9 pieces in any column. Make a trade if possible.* You can let students solve problems on their own using base-ten models and following these rules. You will likely still need to work on the idea of trading. Do not begin written records until students are comfortable and fluent with this process.

Develop the Written Record

The general idea at this stage is to have students record each step *as they do it* with manipulatives. A useful suggestion is to have students turn their lined paper a quarter turn and use the lines to keep the columns separate. For the first few times, guide each step carefully. Most textbooks do quite a good job of illustrating the process.

Another suggestion is to have children work in pairs. One child is responsible for the models and the other for recording the steps. Children reverse roles with each problem.

Figure 6.8 shows a variation of the traditional recording scheme that is quite reasonable, at least for up to three digits. It avoids the little "carried ones" and focuses attention on the value of the digits. If students were permitted to start adding on the left as they are inclined to do, this



Can a trade be made?
How can you tell?

FIGURE 6.7

Setting up the addition algorithm.

recording procedure is the same as that shown for the invented strategy "Add tens, add ones, then combine" (Figure 6.4, p. 166).

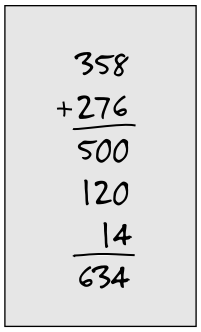


FIGURE 6.8

An alternative recording scheme for addition. Notice that this can be used from left to right as well as from right to left.

The Subtraction Algorithm

The general approach to developing the subtraction algorithm is the same as for addition. When the procedure is completely understood with models, a do-and-write approach connects it with a written form.

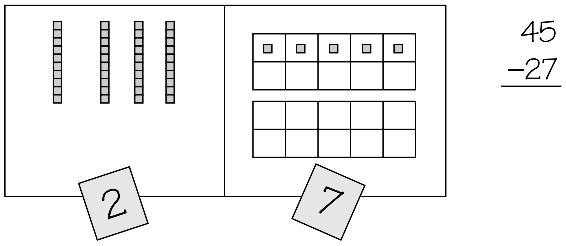
Begin with Models Only

Start by having children model only the top number in a subtraction problem on the top half of their place-value mats. For the amount to be subtracted, have children write each digit on a small piece of paper and place these pieces near the bottom of their mats in the respective columns, as in Figure 6.9. Explain to children that they are to begin working with the ones column first, as they did with addition. To avoid inadvertent errors, suggest making trades before removing any pieces. That way, the full amount on the paper slip can be taken off at once.

Anticipate Difficulties with Zeros

Exercises in which zeros are involved anywhere in the problem tend to cause special difficulties. Give extra attention to these cases while still using models.

The very common error of "borrowing across zero" is best addressed at the modeling stage. For example, in 403 – 138, children must make a double trade, exchanging a hundreds piece for 10 tens and then one of the tens for 10 ones.



Not enough ones to take off 7.
Trade a ten for 10 ones.

FIGURE 6.9

Setting up the subtraction algorithm.

Develop the Written Record

The process of recording each step as it is done is the same as was suggested for addition.

When children can explain symbolism, that is a signal for moving children on to a completely symbolic level. Again, be attentive to problems with zeros.

If students are permitted to follow their natural instincts and begin with the big pieces (from the left instead of the right), recording schemes similar to that shown in Figure 6.10 are possible. The trades are made from the pieces remaining *after* the subtraction in the column to the left has been done. A "borrow across zero" difficulty will still occur, but in problems like this: 462 – 168. Try it.

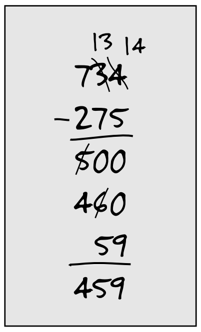


FIGURE 6.10

A left-hand recording scheme for subtraction. Other methods can also be devised.

Assessment Note

It is not unusual for some children in the second grade to use inefficient count-by-ones methods for a long time, showing little or no interest in trying strategies that involve the use of tens. When children do not progress to the more efficient methods that have been illustrated here it can be very disturbing, especially with the pressures of external testing to worry about.

(continued)

A first response should be patience. Many children simply need more time to develop methods that utilize tens. Having said that, it is not sufficient to wait indefinitely with the hope that sooner or later these children will adapt more efficient strategies. There are at least two possible reasons for students' continued use of inefficient methods.

- *Comfort!* Some students are simply not willing to be risk takers. If you have been very accepting of inefficient methods (as you should be), some students develop security with an approach that works for them however tedious it may be and ignore more efficient strategies used by their peers. Perhaps you simply need to challenge these students to find a method that does not require all of that counting or to try a particular tens strategy that was just suggested by one of their classmates.
- *Inadequate base-ten concepts.* In an environment that allows students to build on their own ideas, a student with weak base-ten understanding will not likely use base-ten strategies for computation. This is a more fundamental issue. It suggests that you consider some diagnostic strategies to help decide on next steps.

For some children you may need to find out more about their underlying conceptual understanding of base-ten concepts. It can be useful to stop and listen more carefully to a single child by conducting a short (5- to 10-minute) interview. Often you can do this while others are working on a problem you have given to the class.

Pose a word problem or computation and ask the student to "think out loud" while she works. Explain that you want to find out how she is thinking. Avoid any teaching at this time and make no evaluative comments. Show interest in the student's thinking. Ask questions to clarify thinking that may not be clear. However, if the child is only using count-by-ones methods, you will not be learning what she knows about base-ten ideas. Ask if she can show you how to do the task with base-ten models or the little ten-frame cards. If you feel that there is a real weakness with base-ten ideas, try the more direct diagnoses of base-ten concepts found in Chapter 5 on p. 141.

If you find that your students need more development time with base-ten concepts, consider some of these ideas:

- Present challenges or activities with the hundreds chart. (See Activities 5.11 and 5.12.) Also ask students to solve computation problems using the hundreds chart and focus on those solutions that utilize the rows of ten.
- Do some grouping activities that encourage students to make groups of ten to count. (See Activities 5.7, 5.8, and 5.9.)
- Try activities found in the Chapter 5, "Activities for Flexible Thinking" (p. 145). Look especially at "Numbers, Squares, Sticks, and Dots" (Activity 5.21).
- Have students use the little ten-frame cards to solve the addition computation problems you give them.

Invented Strategies for Multiplication

Computation strategies for multiplication are considerably more complex than for addition and subtraction. Often, but by no means always, the strategies that students invent are very similar to the traditional algorithm. The big difference is that students think about numbers, not digits. They always begin with the large or left-hand numbers.

For multiplication, the ability to break numbers apart in flexible ways is even more important than in addition or subtraction. The distributive property is another concept that is important in multiplication computation. For example, to multiply 43 × 5, one might think about breaking 43 into 40 and 3, multiplying each by 5, and then adding the results. Children require ample opportunities to develop these concepts by making sense of their own ideas and those of their classmates.

Useful Representations

The problem 34 × 6 may be represented in a number of ways, as illustrated in Figure 6.11. Often the choice of a model is influenced by a story problem. To determine how many Easter eggs 34 children need if each colors 6 eggs, children may model 6 sets of 34 (or possibly 34 sets of 6). If the problem is about the area of a rectangle that is 34 cm by 6 cm, then some form of an array is likely. But each representation is appropriate for thinking about 34 × 6 regardless of the context, and students should get to a point where they select ways to think about multiplication that are meaningful to them.

How children represent a product interacts with their methods for determining answers. The groups of 34 might suggest repeated additions—perhaps taking the sets two at a time. Double 34 is 68 and there are three of those, so 68 + 68 + 68. From there a variety of methods are possible.

The six sets of base-ten pieces might suggest breaking the numbers into tens and ones: 6 times 3 tens or 6 × 30 and 6 × 4. Some children use the tens individually: 6 tens make 60. So that's 60 and 60 and 60 (180). Then add on the 24 to make 204.

It is not uncommon to arrange the base-ten pieces in a nice array, even if the story problem does not suggest it. The area model is very much like an arrangement of the base-ten pieces.

All of these ideas should be part of students' repertoire of models for multidigit multiplication. Introduce different representations (one at a time) as ways to explore multiplication until you are comfortable that the class has a collection of useful ideas. At

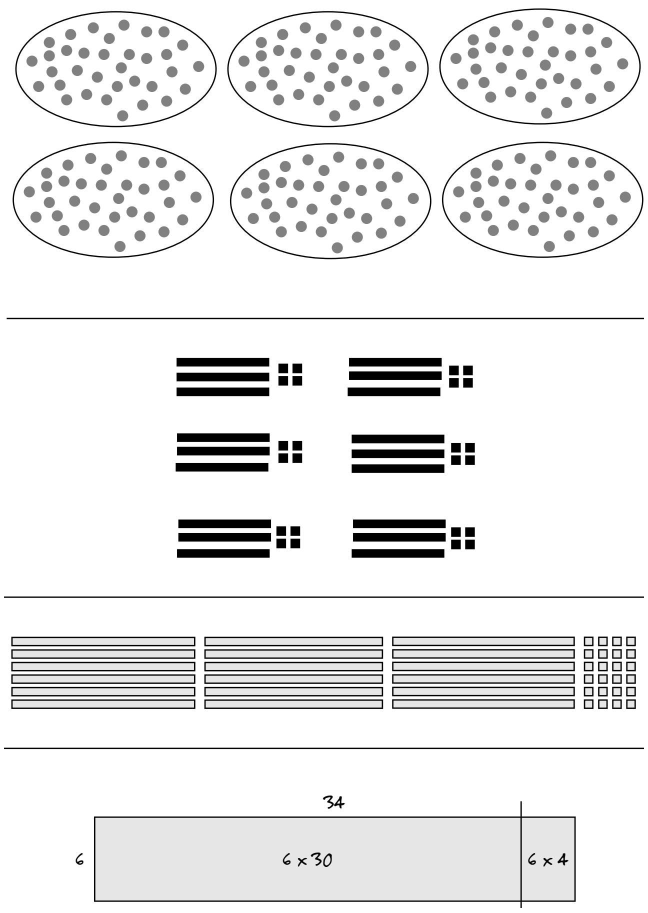


FIGURE 6.11

Different ways to model 34 × 6 may support different computational strategies.

the same time, do not force students who reason very well without drawings to use models when they are not needed.

Multiplication by a Single-Digit Multiplier

As with addition and subtraction, it is helpful to place multiplication tasks in contextual story problems. Let students model the problems in ways that make sense to them. Do not be concerned about mixing of factors (6 sets of 34 or 34 sets of 6). Nor should you be timid about the numbers you use. The problem 3 × 24 may be easier than 7 × 65, but the latter provides challenge. The types of strategies that students use for multiplication are much more varied than for addition and subtraction. However, the following three categories can be identified from the research to date.

Complete-Number Strategies

Children who are not yet comfortable breaking numbers into parts using tens and ones will approach the numbers in the sets as single groups. For students who think this way, Figure 6.12 illustrates two methods they may use. These children will benefit from listening to children who use base-ten models. They may also need more work with base-ten grouping activities where they take numbers apart in different ways.

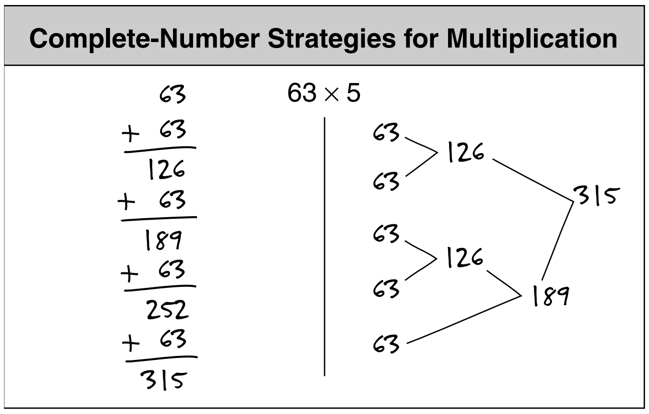


FIGURE 6.12

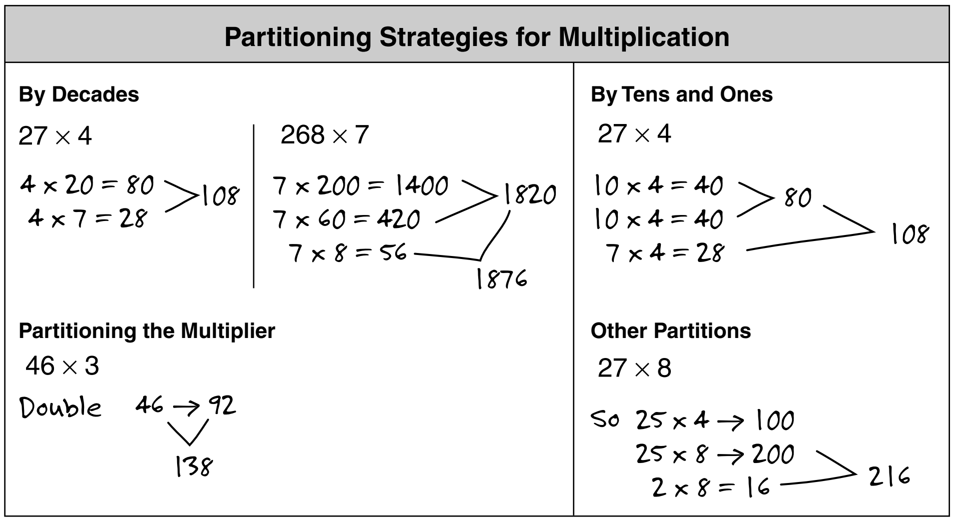
Children who use a complete-number strategy do not break numbers apart into decades or tens and ones.

Partitioning Strategies

Children break numbers up in a variety of ways that reflect an understanding of base-ten concepts, at least four of which are illustrated in Figure 6.13. The "By Decades" approach is the same as the standard algorithm except that students always begin with the large values. It extends easily to three digits and is very powerful as a mental math strategy. Another valuable strat-

FIGURE 6.13

Numbers can be broken apart in different ways to make easier partial products, which are then combined. Partitioning by decades is useful for mental computation and is very close to the standard algorithm.



egy for mental methods is found in the "Other Partitions" example. It is easy to compute mentally with multiples of 25 and 50 and then add or subtract a small adjustment. All partition strategies rely on the distributive property.

Compensation Strategies

Children look for ways to manipulate numbers so that the calculations are easy. In Figure 6.14, the problem 27 × 4 is changed to an easier one, and then an adjustment or compensation is made. In the second example, one factor is cut in half and the other doubled. This is often used when a 5 or a 50 is involved. Because these strategies are so dependent on the numbers involved, they can't be used for all computations. However, they are powerful strategies, especially for mental math and estimation.

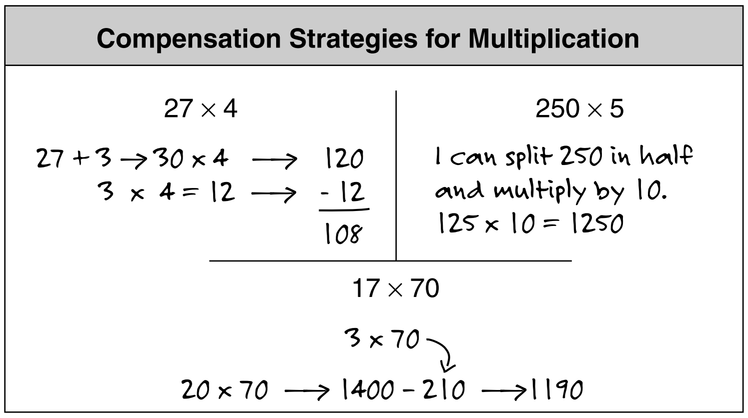


FIGURE 6.14

Compensation methods use a product related to the original. A compensation is made in the answer, or one factor is changed to compensate for a change in the other factor.

Using Multiples of 10 and 100

There is a value in exposing students early to products involving multiples of 10 and 100.

The Scout troop wanted to package up 400 fire starter kits as a fund-raising project. If each pack will have 12 fire starters, how many fire starters are the Scouts going to need?

Children will use 4 × 12 = 48 to figure out that 400 × 12 is 4800. There will be discussion around how to say and write "forty-eight hundred." Be aware of students who simply tack on zeros without understanding why. Try problems such as 3 × 60 or 210 × 4 at grade 3.

The Traditional Algorithm for Multiplication

The traditional multiplication algorithm is probably the most difficult of the four algorithms if students have not had plenty of opportunities to explore their own strategies. Time spent allowing your students to develop a range of invented strategies will pay off in their understanding of the traditional algorithm. While your children are working on multiplication using their invented strategies, be sure to emphasize partitioning techniques, especially those that are similar to the "By Decades" approach shown in Figure 6.13. These strategies tend to be the most efficient and are very close to the traditional algorithm. In fact, students who are using one or more partitioning strategies with a one-digit multiplier have no real need to learn any other approach.

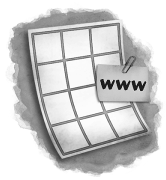
The traditional algorithm can, like the "By Decades" approach, be developed with either a repeated addition model or an area model. For single-digit multipliers, the difference is minimal. It is only for two-digit multipliers that the area model has some advantages.

An Area Model Development

We will briefly explain a development of the traditional algorithm using an area model because there is some advantage to this approach when students move to two-digit multipliers. As with all prescribed algorithms, you will have to direct your students rather carefully rather than utilize a full problem-oriented approach.

Give students a drawing of a rectangle 47 cm by 6 cm. *How many small square centimeter pieces will fit in the rectangle?* (What is the area of the rectangle in square centimeters?) Let students solve the problem in groups before discussing it as a class.

As shown in Figure 6.15, the rectangle can be "sliced" or separated into two parts so that one part will be 6 ones by 7 ones, or 42 ones, and the other will be 6 ones by 4 tens, or 24 tens. Notice that the base-ten language "6 ones times 4 tens is 24 tens" tells how many *pieces* (sticks of ten) are in the big section. To say "6 times 40 is 240" is also correct and tells how many units or square centimeters are in the section. Each section is referred to as a *partial product*. By adding the two partial products, you get the total product or area of the rectangle.



BLM 19

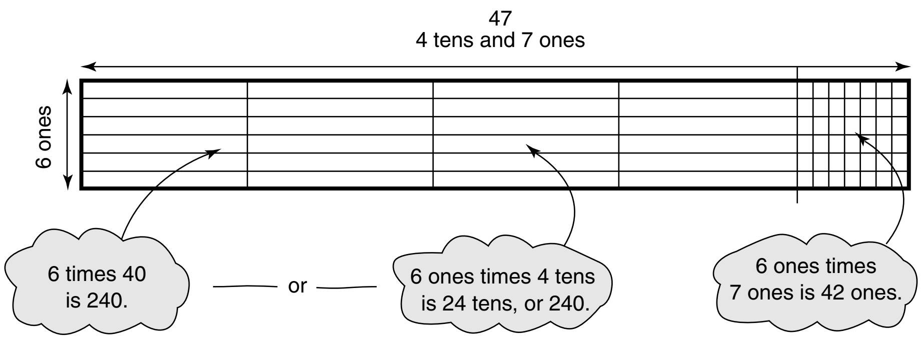
To avoid the tedium of drawing large rectangles and arranging base-ten pieces, use the base-ten grid paper found in the Blackline Masters. On the grid paper, students can easily draw accurate rectangles showing all of the pieces. Check to be sure students understand that for a product such as 74 × 8, there are two partial products, 70 × 8 = 560 and 4 × 8 = 32, and the sum of these is the product. Do not force any recording technique on students until they understand how to use the two dimensions of a rectangle to get a product.

Develop the Written Record

When the two partial products are written separately as in Figure 6.16(a), there is little new to learn. Students simply record the products and add them together. As illustrated, it is possible to teach students how to write the first product with a carried digit so that the combined product is written on one line. This traditional recording scheme is known to be problematic. The little carried digit is often the source of difficulty—it gets added in before the second multiplication or is forgotten.

FIGURE 6.15

A rectangle filled with base-ten pieces is a useful model for two-digit-by-one-digit multiplication.



There is absolutely no practical reason why students can't be allowed to record both partial products and avoid the errors related to the carried digit. When you accept that, it makes no difference in which order the products are written. Why not simply permit students to do written multiplication as shown in Figure 6.16 without carrying? Furthermore, that is precisely how this is done mentally.

Most standard curricula progress from two digits to three digits with a single-digit multiplier. Students can make this progression easily. They still should be permitted to write all three partial products separately and not have to bother with carrying.

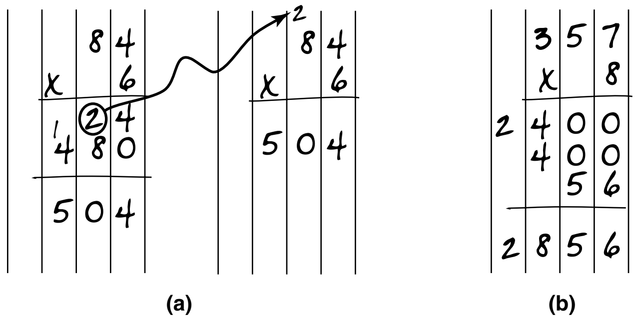


FIGURE 6.16

(a) In the standard form, the product of ones is recorded first. The tens digit of this first product can be written as a "carried" digit above the tens column. (b) It is quite reasonable to abandon the carried digit and permit the partial products to be recorded in any order.

Two-Digit Multipliers

With the area model, the progression to a two-digit multiplier is relatively straightforward. Rectangles can be drawn on base-ten grid paper, or full-sized rectangles can be filled in with base-ten pieces. There will be four partial products, corresponding to four different sections of the rectangle. Figure 6.17 shows a rectangle partitioned into the usual partial products.

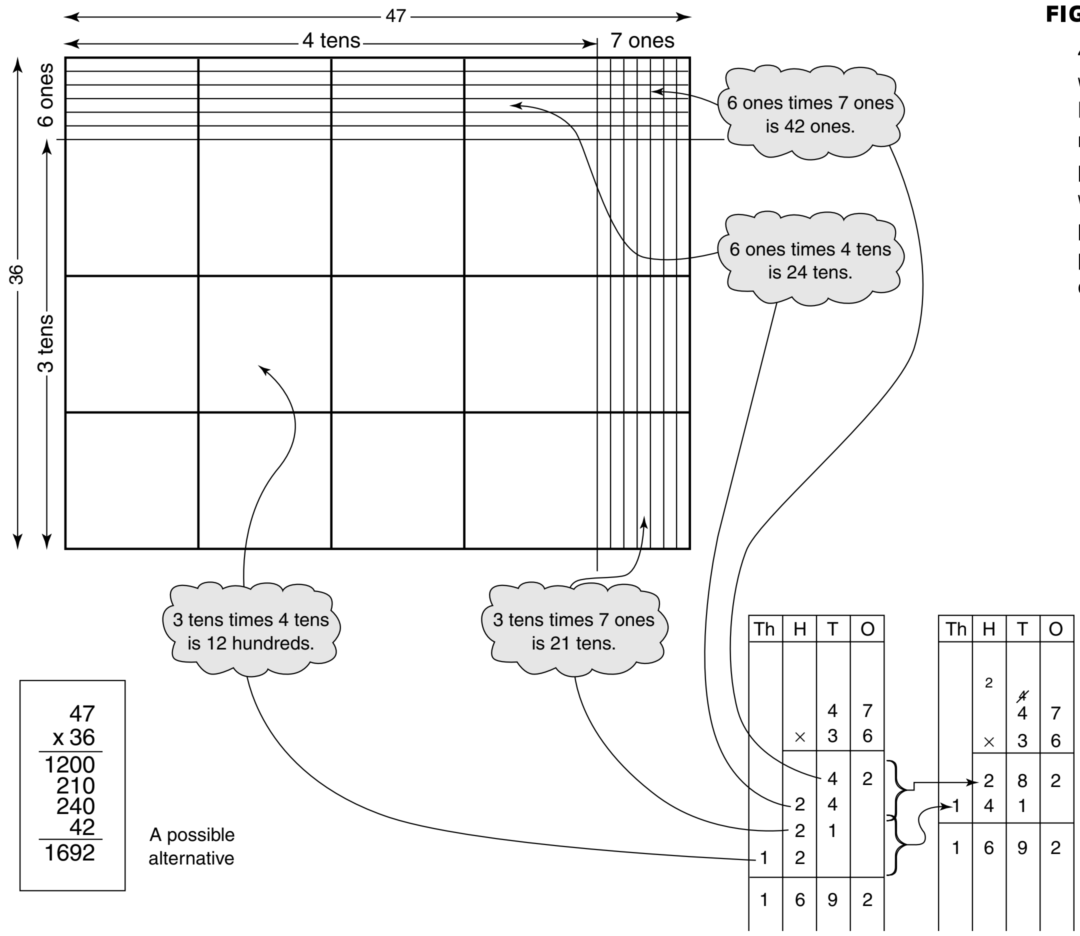


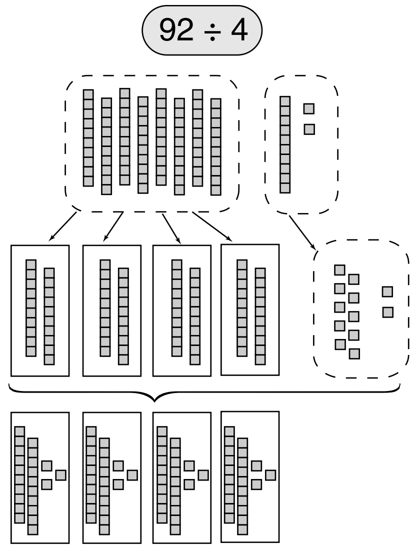
FIGURE 6.17

47 × 36 rectangle filled with base-ten pieces. Base-ten language connects the four partial products to the traditional written format. Note the possibility of recording the products in some other order.

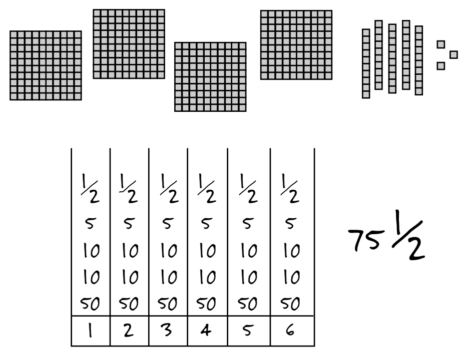
Invented Strategies for Division

In our discussion of division facts (Chapter 4), we included something we called "near facts." In a near fact, the divisor and quotient are both less than ten but there is a remainder, as in 44 ÷ 8. Third-grade students should have ample experiences with near facts. When these problems are expanded to those in which the quotients are more than 9 (e.g., 73 ÷ 6), the process evolves into invented strategies for division. Third grade is not too soon for students to begin exploring division strategies, but there is little need to teach the standard algorithm.

(a)



(b) 453 ÷ 6
(share with 6 kids)



(c) 143 jelly beans shared with 8 kids

Try 14 × 8 → 112
12 groups of 8 is 96.
12 groups in 100 leaves 4.
5 groups of 8 is 40.
And 3 more left over.
12 + 5 is 17 with 7 left.

FIGURE 6.18

Students use both models and symbols to solve division tasks.

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Sharing and Measurement Problems

Recall that there are two concepts of division. First is the partition or fair-sharing idea, illustrated by this story problem:

The bag has 783 jelly beans, and Laura and her four friends want to share them equally. How many jelly beans will Laura and each of her friends get?

Then there is the measurement or repeated subtraction concept:

Jumbo the elephant loves peanuts. His trainer has 625 peanuts. If he gives Jumbo 20 peanuts each day, how many days will the peanuts last?

Students should be challenged to solve both types of problems. However, the fair-share problems are often easier to solve with base-ten pieces. Furthermore, the traditional algorithm is built on this idea. Eventually, students will develop strategies that they will apply to both types of problems, even when the process does not match the action of the story.

Figure 6.18 shows some strategies that fourth-grade children have used to solve division problems. The first example illustrates 92 ÷ 4 using base-ten pieces and a sharing process. A ten is traded when no more tens can be passed out. Then the 12 ones are distributed, resulting in 23 in each set. This direct modeling approach with base-ten pieces is quite easy even for third-grade students to understand and use.

In the second example, the student sets out the base-ten pieces and draws a "bar graph" with six columns. After noting that there are not enough hundreds for each kid, he mentally splits the 3 hundreds in half, putting 50 in each column. That leaves him with 1 hundred, 5 tens, and 3 ones. After trading the hundred for tens (now 15 tens), he gives 20 to each, recording 2 tens in each bar. Now he is left with 3 tens and 3 ones, or 33. He knows that 5 × 6 is 30, so he gives each kid 5, leaving him with 3. These he splits in half and writes $\frac{1}{2}$ in each column.

The child in the third example is solving a sharing problem but tries to do it as a measurement process. She wants to find out how many 8s are in 143. Initially she guesses. By multiplying 8 first by 10, then by 20, and then by 14, she knows the answer is more than 14 and less than 20. After some more work (not shown), she rethinks the problem as how many 8s in 100 and how many in 40.

Missing Factor Strategies

You can see in Figure 6.18 how the use of base-ten pieces tends to lead to a digit-by-digit strategy—share the hundreds first, then the tens, then the ones. Although this is precisely the conceptual background behind the traditional algorithm, it is digit oriented as opposed to an approach that helps students think of the whole value of the dividend. In Figure 6.18(c), the student is using a multiplicative approach. She is trying to find out, "What number times 8 will be close to 143 with less than 8 left over?" This is a good method to suggest to students in grade 3 or 4. It will build on their multiplication skills, it is a method that lends itself to mental estimation, and it can work quite well for most purposes.

STOP **Before reading further, consider the task of determining the quotient of 318 ÷ 7 by trying to figure out *what number times 7* (or *7 times what number*) is close to 318 without going over. Do not use the standard algorithm.**

There are several places to begin solving this problem. For instance, since 10 × 7 is 70 and 100 × 7 is 700, it has to be between 10 and 100, probably closer to 10. You might start adding up 70s:

70
\+ 70 is 140
\+ 70 is 210
\+ 70 is 280
\+ 70 is 350

So four 70s is not enough and five is too much. It has to be forty-something. At this point you could guess at numbers between 40 and 50. Or you might add on 7s. Or you could notice that forty 7s (280) leaves you with 20 plus 18 or 38. Oh—five 7s will be 35 of the 38 with 3 left over. In all, that's 40 + 5 or 45 with a remainder of 3.

This missing-factor approach is likely to be invented by some students if they are solving measurement problems such as the following:

Grace can put 6 pictures on one page of her photo album. If she has 82 pictures, how many pages will she need?

Alternatively, you can simply pose a task such a 82 ÷ 6 and ask students, "What number times 6 would be close to 82?" and continue from there.

The Traditional Algorithm for Division

Long division is the one traditional algorithm that starts with the left-hand or big pieces. The conceptual basis for the algorithm most often taught in textbooks is the partition or fair-share method.

Typically, the division algorithm with one-digit divisors is introduced in the third grade. If done well, it should not have to be retaught.

Begin with Models

Return again to the first example in Figure 6.18. The student is using base-ten models to solve 92 ÷ 4. Notice that the first step is to share the tens. *There are 9 tens. How many tens can be put in each of the 4 groups?* The tens are distributed, 2 per group, and then there is only one left. This last ten is exchanged for 10 ones, making a total of 12 ones. Since 4 × 3 is 12, 3 ones can be put in each group with none left over.

If students use a pregrouped base-ten model such as those shown in Figure 6.18, they will need to understand the concept of trading a ten for 10 ones (and later a hundred for 10 tens). This is not difficult but may require some extra work.

STOP **Try the distributing or sharing process yourself using base-ten pieces (or draw squares, sticks, and dots). Use the problem 524 ÷ 3. Try to talk through the process without using "goes into." Think sharing.**

Develop the Written Record

The recording scheme for the long-division algorithm is not completely intuitive. You will need to be quite directive in helping children learn to record the fair sharing with models. There are essentially four steps:

1. *Share* and record the number of pieces put in each group.
2. *Record* the number of pieces shared in all. Multiply to find this number.
3. *Record* the number of pieces remaining. Subtract to find this number.
4. *Trade* (if necessary) for smaller pieces and combine with any that are there already. Record the new total number in the next column.

When students model problems with a one-digit divisor, steps 2 and 3 seem unnecessary. Explain that these steps really help when you don't have the pieces there to count.

Figure 6.19 details each step of the recording process just described. On the left, you see the traditional algorithm. To the right is a suggestion that matches the actual action with the models by explicitly recording the trades. Instead of the somewhat mysterious "bring-down" procedure, the traded pieces are crossed out, as is the number

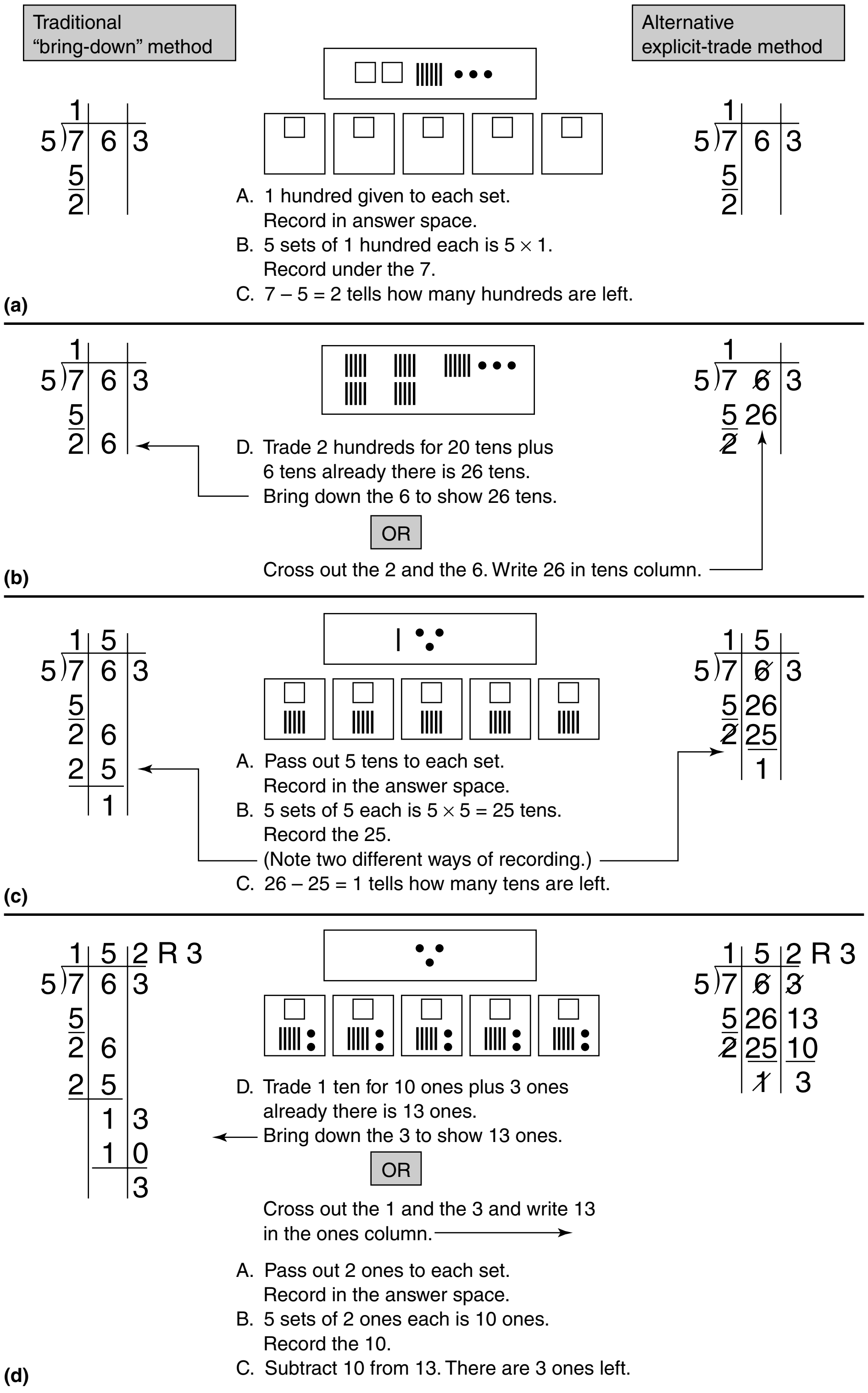


FIGURE 6.19

The traditional and explicit-trade methods are connected to each step of the division process. Every step can and should make sense.

of existing pieces in the next column. The combined number of pieces is written in this column using a two-digit number. In the example, 2 hundreds are traded for 20 tens, combined with the 6 that were there for a total of 26 tens. The 26 is therefore written in the tens column.

Students who are required to make sense of the long-division procedure find the explicit-trade method easier to follow. It is important to spread out the digits in the dividend when writing down the problem. Lines separating place-value columns are strongly recommended. (The explicit-trade method is a Van de Walle invention. It has been used successfully in grades 3 to 8. You will not find it in textbooks.)

Both the explicit-trade method and the use of place-value columns will help with the problem of leaving out a middle zero in a problem (see Figure 6.20).

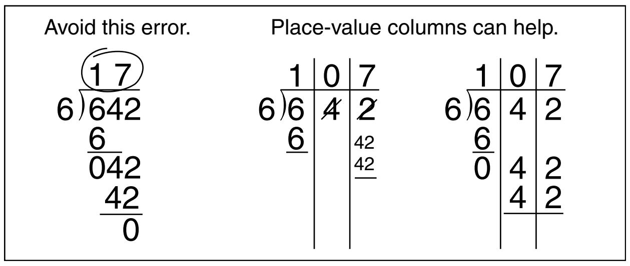


FIGURE 6.20

Using lines to mark place-value columns can help avoid forgetting to record zeros.

Assessment Note

Parents are perhaps more interested in their children's computational skills than in any other area. When students do well on computation tests, parents are pleased. But what do you know when students do not do well? At best you can make inferences based on the papers turned in. You can look for basic-fact errors and carelessness or perhaps find a systematic error in an algorithm. What you do not know is how children are solving these problems and what ideas and strategies they have developed that are useful or need further development.

When computational strategies and algorithms are developed in the manner suggested in this chapter, every day you are presented with a wealth of assessment data. The important thing is to gather, record, and use these data for individual children the same as you would for tests and quizzes. A simple chart something like the one in Figure 6.21 may be all you need. Note that the third

FIGURE 6.21

A checklist with space for comments or notes lets you record daily observations of students' direct modeling and invented strategies.

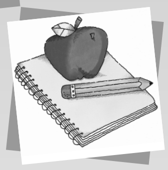
Topic: Mental addition and subtraction

Student	Adds 2-digit + 1-digit numbers	Adds 2-digit + 2-digit numbers; note methods	Flexibility in choosing a method: 1, 2, 3	Comments
Lalie				
Pete				
Sid				
Lakeshia				

column includes a minirubric or a three-point scale. Students' names can be arranged in groups, by how they sit in the room, or alphabetically—any way that makes them easy to find.

As you walk around in the during portion of your lessons, and also in the after portion when children explain their computation strategies and reasoning, you can make notes on the chart. Make a new chart each week but keep the old ones to provide evidence of growth over time. These charts can be useful for grading and for parent conferences. There is no harm in giving an occasional quiz or test of computational skills. But avoid giving more value to tests simply because they are objective.

EXPANDED LESSON



Exploring Subtraction Strategies

GRADE LEVEL: Second or third grade.

MATHEMATICS GOALS

- To develop flexible strategies for subtracting two-digit numbers with an emphasis on adding-up methods.
- To promote the use of tens in computational strategies.

THINKING ABOUT THE STUDENTS

The students have been working on a variety of invented strategies for adding two-digit numbers. It is not necessary that this skill be mastered before beginning subtraction. This lesson may or may not be the first subtraction lesson. The assumption is that students have not been taught the traditional algorithms for addition or subtraction.

MATERIALS AND PREPARATION

Prepare two story problems either on a transparency or duplicate on a sheet of paper. If duplicated, leave half of the page for each problem.

lesson

BEFORE

The Task

- Provide students with two story problems either on paper or on the overhead. Here are two possibilities:

  David's book has 72 pages. He has already read 35 pages. How many more pages does David have to read to finish his book?

  Tara keeps her books on two bookshelves. She has 24 books in all. On the top shelf she has 16 books. How many books does Tara have on the bottom shelf?

  Students are to solve each problem using any method that they want.

Brainstorm

- Read the first problem together with the students. Have them think how they would go about solving the problem. What would they do first?
- Give students some time to think of a way to solve the problem and then call on several students to share their ideas. Try to elicit some details. For example, if a student says, "I would start with 35 and add up to 72," ask what she would add on first. You want to give students who do not know where to begin some good ideas. Ask other students if they would begin differently. It is not sufficient to say, "I would subtract." If students use a take-away strategy or even a counting-by-ones approach, that is okay. Do not force any method.

Establish Expectations

- Students are to show how they solved each problem. They should provide enough information so that if someone picked up their paper they would be able to tell how they got the answers. Remind students that they will be sharing their ideas with the class.

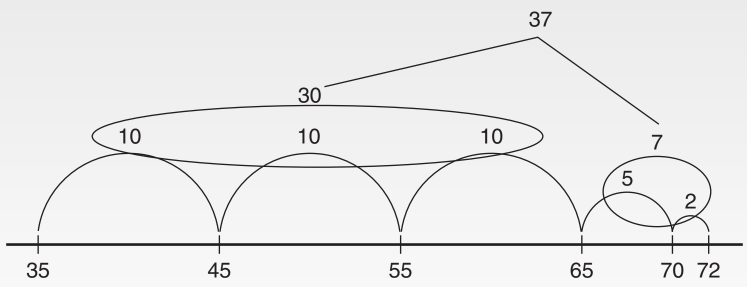
DURING

- Monitor students' work. For students who are stuck, ask them to tell you what ideas they have thought about. Try to make a suggestion that builds on the student's ideas. For example, if the student says, "I want to take 35 from 72," you might suggest that the student try taking away 10 at a time. After making a suggestion, walk away. Do not be overly guiding.
- Watch for students who solve the problem but have not written enough to explain their solution. Ask them about their method and then require them to write some more to explain what they have just told you. You may have to help some students with ways to record their ideas.

- Do not correct students who have made errors.
- Watch for students to share their methods. If usually reluctant students have a solution, advise them that you may call on them to share their ideas.
- Challenge early finishers with a three-digit problem: 314 – 197.

AFTER

- When all students have completed their work, ask for answers to the first problem. Record these on the board. Pick one of the answers and ask, *Who got this answer?* Select a student to share his or her method.
- As the students explain their thinking, try to record their process on the board. Avoid aligning the two numbers vertically as in the traditional algorithm. One good strategy is to use a blank number line. As students explain what they did, you can indicate each portion on the number line, recording each jump as they explain. For example, suppose that a student starts with 35 and adds 10 and 10 and 10 (to get to 65), then adds 5 (to get to 70), and then 2 more. Your record on the board may look like this:



72 – 35 = 37

- For each jump on the number line, indicate the size of the jump. For adding-up strategies, the answer will usually be the sum of the jumps. For take-away strategies, the answer will be where the jumps end.
- When more than one answer has been offered, be sure to have a method shared for each answer. By asking the class if they agree with, understand, or have questions for the student who is sharing, the responsibility for deciding what is correct falls to the class, not you. In any case, try to get several solution methods for the problem.
- Have students share their solution strategies for the second problem in a similar manner.

ASSESSMENT NOTES

- Watch for students who are solving these problems by counting by ones. Some may even count both numbers by ones rather than count on. Others may make tallies or use counters for the larger number and then mark off or remove the subtrahend.
- Try to keep track of students who are using a take-away method and those who use an adding-on approach. For some problems, take-away strategies are often more difficult.

next steps

- Students who are not using tens at all (counting by ones) will benefit by solving similar problems using the little ten-frame cards. (See Activity 5.26, p. 149.) Also consider other activities from the "Activities for Flexible Thinking" section of Chapter 5.
- If few or none of your students used an adding-on strategy, try Activity 5.23, "The Other Part of 100" and Activity 5.24, "Compatible Pairs."
- Even if students are successful with these problems, it is appropriate to provide a lot of additional practice conducted in a similar manner to this lesson. It is not necessary to always use story problems, nor is it necessary to restrict the tasks to two-digit numbers. As students become more proficient, challenge those who are able to solve problems such as these mentally.