PART ONE

Misconceptions and Error Patterns

This book is designed to help us improve mathematics instruction in our classrooms by becoming more diagnostically oriented. Diagnosis should be continuous throughout instruction.

Why do our students sometimes learn misconceptions and erroneous procedures when learning to compute? How important is it to teach paper-and-pencil computation procedures in our age of calculators and computers? Part One addresses these questions, and the need for conceptually oriented instruction.

As we teach our students, we need to be alert to misconceptions and error patterns they may learn. Part One provides many opportunities to identify misconceptions and error patterns in student papers and to think about why these students may have used the procedures they did. You learn what might be done to help students who are experiencing such difficulties.

In Chapters 2 through 8, student papers are presented so you can study them and infer what the student was actually thinking and doing when completing the paper. Then you turn to a page where the difficulty is discussed, and you have an opportunity to think about instructional activities that may help the student. Suggested activities are described and you can compare your ideas with those of the author.

Some papers include a few correct answers even though the student’s thinking will not always lead to a correct result. When this happens, students are encouraged to believe that their thinking is correct and their procedure is satisfactory.

CHAPTER 1

Computation, Misconceptions, and Error Patterns

Number and Operations Standard

Instructional programs from prekindergarten through grade 12 should enable all students to—

- *understand numbers, ways of representing numbers, relationships among numbers, and number systems;*
- *understand meanings of operations and how they relate to one another;*
- *compute fluently and make reasonable estimates.*[1]

In this age of calculators and computers, do our students actually need to learn paper-and-pencil procedures? We want our students to understand mathematical concepts and to compute fluently, but how does this relate to students learning to do paper-and-pencil procedures when calculators are so readily available?

As we examine these and other questions in this chapter, we will find that even in our technological age, paper-and-pencil computation is often needed. True, paper-and-pencil procedures constitute only one alternative for computing—though it often makes sense to use such procedures. It is also true that while our students are learning to compute with paper and pencil, their knowledge of basic facts, numeration concepts, and various principles can be further developed—knowledge needed for doing other forms of computation.

Instruction in Mathematics

Our society is drenched with data. We have long recognized that verbal literacy is essential to our well-being as a society; now we recognize that quantitative literacy or *numeracy* is also essential.

Accordingly, our goals are changing. We want to see instructional programs enable students to understand and use mathematics in a technological world.

We are not interested in students just doing arithmetic in classrooms; we want to see the operations of arithmetic applied in real-world contexts where students observe and organize data. We no longer assume that students must be skillful with computation before they can actually begin investigating interesting topics in mathematics.

Instruction in mathematics is moving toward covering fewer topics but in greater depth and toward making connections between mathematical ideas. Increasingly, mathematics is being perceived as a science of patterns rather than a collection of rules. In truth, there are those who characterize algebra as *generalized* arithmetic, and there are those who even propose that ". . . the teaching and learning of arithmetic be conceived as the foundation for algebra."[2]

Number and Operations is only one of the five content standards for grades pre-K through 12 in *Principles and Standards for School Mathematics,* published in 2000 by the National Council of Teachers of Mathematics (NCTM). But computation, including the basic facts of arithmetic, is often involved when the other four content standards are learned and applied: Algebra, Geometry, Measurement, and Data Analysis and Probability. Moreover, application in every grade of the five process standards frequently entails the basic facts of arithmetic and computation: Problem Solving, Reasoning and Proof, Communication, Connections, and Representation. The basic facts and different methods of computation are very much a part of standards-based instruction in mathematics today.

The computations of arithmetic are not being ignored. The importance of computation is made clear in *Principles and Standards for School Mathematics.*

> **Knowing basic number combinations—the single-digit addition and multiplication pairs and their counterparts for subtraction and division—is essential. Equally essential is computational fluency—having and using efficient and accurate methods for computing.[3]**

Number and Operations were later highlighted for grades pre-K through 8 by NCTM in their *Curriculum Focal Points.*

Computational Fluency

If our students are to have computational fluency, if they are to have and use efficient and accurate methods for computing, they need conceptual understanding—"comprehension of mathematical concepts, operations, and relations" and procedural fluency—"skill in carrying out procedures flexibly, accurately, efficiently, and appropriately." Both are aspects of mathematical proficiency as defined by the Mathematics Learning Study Committee of the National Research Council.[4]

Increasingly we need to integrate arithmetic and all of the mathematics we teach with the world of our students, including their experiences with other subject areas. In order to solve problems encountered in the world around them, our students need to know not only how to compute a needed number, but also *when* to compute. In order for them to know when to use specific operations, we need to emphasize the meanings of operations during instruction.

Furthermore, in order for our students to gain computational fluency, they need to be able to use different methods of computation in varied problem-solving situations.

> **Part of being able to compute fluently means making smart choices about which tools to use and when. Students should have experiences that help them learn to choose among mental computation, paper-and-pencil strategies, estimation, and calculator use. The particular context, the question, and the numbers involved all play roles in those choices.**[5]

If we focus on paper-and-pencil procedures but do not introduce other methods of computing, our students are apt to believe that the process of computing is limited to paper-and-pencil procedures.

When students have a problem to solve, there are decisions to be made before any required computation begins. Consider this part of a conversation overheard in a student math group led by Chi. Calculators were available and students were free to use one if they needed it, but the teacher also encouraged other methods of exact computation: mental computation and paper-and-pencil procedures.

CHI:	We have three word problems to solve. Here's Problem A. *You are to help the class get ready for art class. There are 45 pounds of clay for 20 people. How many pounds do you give each person?*
RAUL:	We need an exact answer. Each person should get the same amount.
TERRY:	I don't think we need to use the calculator. Each person gets a little more than two pounds . . . but how much more?
SONJA:	That's easy, just divide 45 by 20. The students proceed to divide 45 by 20 with paper and pencil.
CHI:	Here's Problem B. *Wanda's scores for three games of darts are 18, 27, and 39. What is her average score?*
RAUL:	Another exact answer.
CHI:	Shall we get the calculator?
SONJA:	I can do it in my head.
TERRY:	I don't see how. I'm going to use paper; it's easy.
SONJA:	Round each number up . . . 20 + 30 + 40 is 90 . . . divided by 3 is 30. But we need to subtract: 2 and 3 (that's 5) and one more . . . 6 divided by 3 is 2. Two from 30 is 28. Her average score is 28.
TERRY:	That's what I got, too.

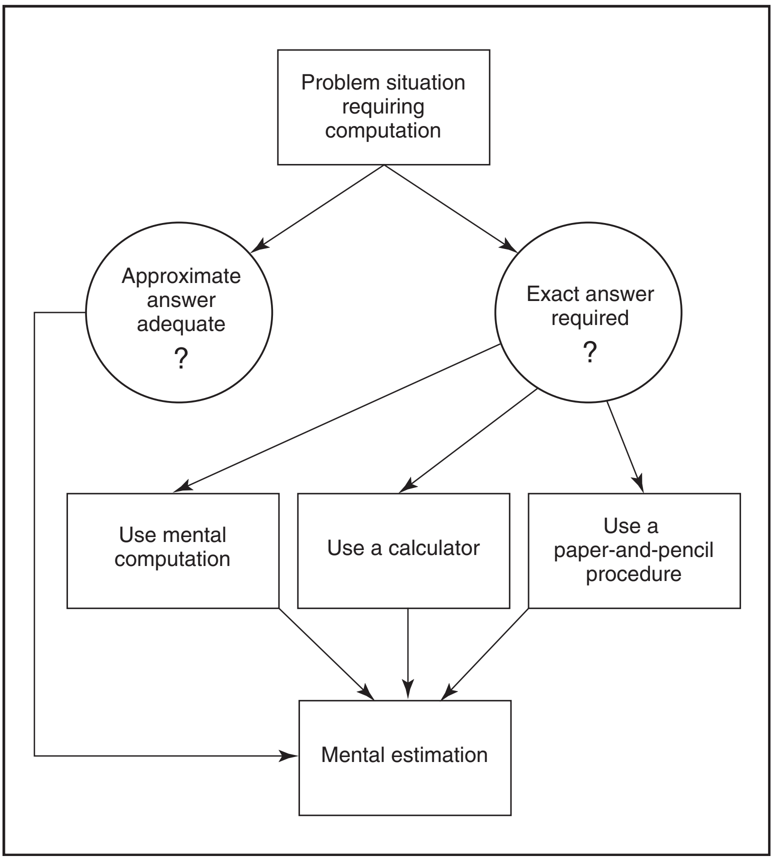
Different methods of computation are listed in Figure 1.1. Because an approximation is often sufficient, the first decision a student must make is whether an exact number is needed. In regard to exact computation, paper-and-pencil procedures constitute only one of the methods of computation available.

FIGURE 1.1 Methods of computation.

METHODS OF COMPUTATION

Approximation	Exact Computation
– Mental estimation	– Mental computation
	– Calculator
	– Paper-and-pencil

FIGURE 1.2 Decisions about calculation in problem situations requiring computation.



Source: Based on a chart in *Curriculum and Evaluation Standards for School Mathematics* (Reston, VA: National Council of Teachers of Mathematics, 1989), p. 9.

The diagram in Figure 1.2 focuses on decisions about the method of computation to be used for solving a particular problem. The actual teaching of different methods of computation is addressed in Chapter 10.

When a solution to $300 - 25 = \square$ is needed by a fourth grader, mental computation is likely the most appropriate method to use. But there are times when a paper-and-pencil strategy is the most efficient procedure for an individual. When a calculator is not immediately available and an exact answer is needed for the sum of two or three multi-digit numbers, it often makes sense to use a paper-and-pencil procedure.

Because computational procedures are tools for helping us solve problems, whenever possible the *context* for teaching different methods of computation should be a problem-solving situation; we need to keep focused on problem solving as we teach computation procedures. The *goal* of instruction in computation today continues to be computational fluency. Students need to be able to use efficient and accurate methods for computing if they are to enjoy success in most areas of mathematics.

Algorithms

It is common to speak of algorithmic *thinking,* which uses specific step-by-step procedures, in contrast to thinking, which is more self-referential and recursive. Polya's four-step model for problem solving is an example of algorithmic thinking.[6]

An algorithm is a step-by-step procedure for accomplishing a task, such as solving a problem. In this book, the term usually refers to paper-and-pencil procedures for finding a sum, difference, product, or quotient. The paper-and-pencil procedures that individuals learn and use differ over time and among cultures. If a "standard" algorithm is included in your curriculum, remember that curriculum designers made a choice. If some students have already learned different algorithms (for example, a different way to subtract learned in Mexico or in Europe), remember that these students' procedures are quite acceptable if they always provide the correct number.

Usiskin lists several reasons for teaching various types of algorithms, a few of which follow. These apply to paper-and-pencil procedures as well as to the other types of algorithms he discusses.[7]

- *Power.* An algorithm applies to a class of problems (e.g., multiplication with fractions).
- *Reliability and accuracy.* Done correctly, an algorithm always provides the correct answer.
- *Speed.* An algorithm proceeds directly to the answer.
- *A record.* A paper-and-pencil algorithm provides a record of how the answer was determined.
- *Instruction.* Numeration concepts and properties of operations are applied.

As we teach paper-and-pencil procedures, we need to remember that our students are learning and applying concepts as they are learning procedures.

Conceptual Learning and Procedural Learning

The importance of conceptual learning is stressed by NCTM in *Principles and Standards for School Mathematics.* Conceptual learning in mathematics always focuses on ideas and on generalizations that make connections among ideas. In contrast, procedural learning focuses on skills and step-by-step procedures.[8]

Sadly, procedures are sometimes taught without adequately connecting the steps to mathematical ideas. Both conceptual learning and procedural learning are necessary, but procedural learning needs to be tied to conceptual learning and to real-life applications. Procedural learning *must* be based on concepts already learned. There is evidence that learning rote procedures before learning concepts and how they are applied in those procedures actually interferes with later meaningful learning.[9]

In order for concepts to build on one another, ideas need to be understood and woven together. As a part of their increasing number sense, our younger students need to understand principles and concepts associated with whole numbers and numerals for whole numbers. Then, students begin to make reasonable estimates and accurate mental computations.

Students need to understand the meaning of each operation (and not just do the computations), so they can decide which operation is needed in particular situations. Otherwise, they do not know which button to push on a calculator or which paper-and-pencil procedure to use.

Conceptual understanding is *so* important that some mathematics educators stress the invention of algorithms by young students; they fear that early introduction of standard algorithms may be detrimental and not lead to understanding important concepts.[10] Understanding the concepts and reasoning involved in an algorithm does lead to a more secure mastery of that procedure. It is also true that standard algorithms *can* be taught so that students understand the concepts and reasoning associated with specific procedures.

Paper-and-pencil procedures that we teach actually involve more than procedural knowledge; they entail conceptual knowledge as well. Many of the instructional activities described in this book are included because students need to understand specific concepts. Our students are not merely mechanical processors; they are involved conceptually as they learn—even when we teach procedures.

> **[I]nstruction can emphasize conceptual understanding without sacrificing skill proficiency . . . understanding does not detract from skill proficiency and may even enhance it.**[11]

It has long been recognized that instruction should balance conceptual understanding and skill proficiency. One of the classic publications of mathematics education is William Brownell's "Meaning and Skill—Maintaining the Balance," published originally in 1956 but reprinted twice by the National Council of Teachers of Mathematics, once as recently as 2003.[12]

It must be recognized that as a student uses a specific paper-and-pencil procedure over time, it becomes more automatic. The student employs increasingly less conceptual knowledge and more procedural knowledge, a process researchers sometimes call "proceduralization."

Paper-and-Pencil Procedures Today

Even with calculators and computers available, our students need to acquire skill with paper-and-pencil procedures. Writing in *Educational Leadership,* Loveless and Coughlan conclude:

> **We would simply like all students to learn how to add, subtract, multiply, and divide using whole numbers, fractions, and decimals—and accurately compute percentages—by the end of 8th grade. Only by mastering these skills will students have the opportunity to learn higher-level mathematics.**[13]

NCTM's *Principles and Standards for School Mathematics* clearly supports teaching computation skills:

> **[S]tudents must become fluent in arithmetic computation—they must have efficient and accurate methods that are supported by an understanding of numbers and operations. "Standard" algorithms for arithmetic computation are one means of achieving this fluency.**[14]

Although needed arithmetic computation skills include estimation, mental computation, and using calculators, it is noteworthy that our students also need to be able to use appropriate paper-and-pencil algorithms when it makes sense to do so.

Students sometimes learn error patterns as we teach these procedures. We can teach diagnostically and carefully observe what our students do, looking for misconceptions and error patterns in their written work.

Learning Misconceptions and Error Patterns

All of us, including our students, make mistakes from time to time. Some individuals suggest that if you don't make mistakes, you are probably not working on hard enough problems—and *that's* a big mistake.

However, there is a difference between the careless mistakes we all make, and *mis*conceptions about mathematical ideas and procedures. Students learn concepts, and sometimes they also learn misconceptions—in spite of whatever we try to teach them. Error patterns in computation often reveal misconceptions our students have learned.

The mathematical ideas and procedures (or rules) a student learns may be correct or they may be full of misconceptions, but the *process* of learning those ideas and procedures is basically the same. During experiences with a concept or a process (or a procedure), a student focuses on whatever the experiences appear to have in common and connects that information to information already known.

Consider the student whose only school experiences with the number idea we call *five* involve manila cards with black dots in the familiar domino pattern (Figure 1.3a). That student may draw from those experiences a notion of five that includes many or all of the characteristics his experiences had in common: possibly black on manila paper, round dots, or a specific configuration. One of the author's own students, when presenting to her students the configuration associated with Stern pattern boards (Figure 1.3b) was told, "That's not five. Five doesn't look like that."

More young students will name as a triangle the shape in Figure 1.3c than the shape in Figure 1.3d; yet both are triangles. Again, configuration (or even the orientation of the figure) may be a common characteristic of a child's limited range of experiences with triangles.

Dr. Geoffrey Matthews, who organized the Nuffield Mathematics Teaching Project in England, told about a child who computed correctly one year but missed half of the problems the next year. As the child learned to compute, he adopted the rule, "Begin on the side by the piano." The next school year the child was in a room with the piano on the other side, and he was understandably confused about where to start computing.

Concept cards, which are often used in learning centers, also illustrate concept formation (Figure 1.4). As a student examines a concept card, he sees a name

FIGURE 1.3 Patterns for five and triangles.

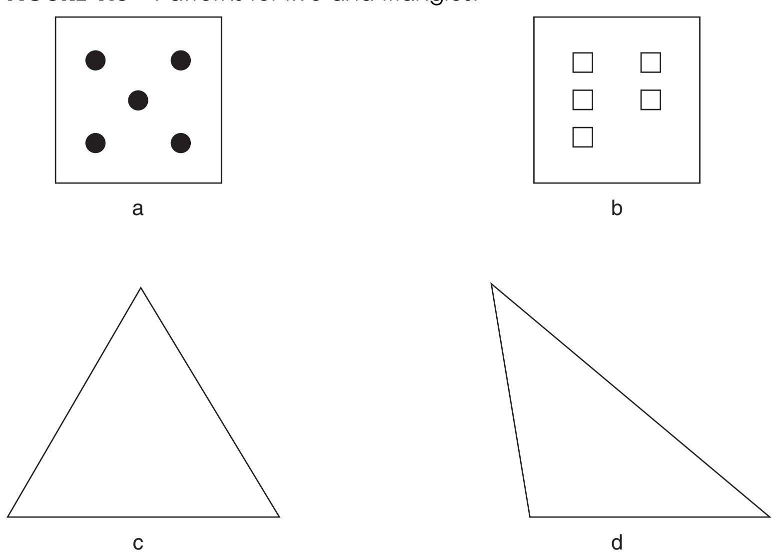
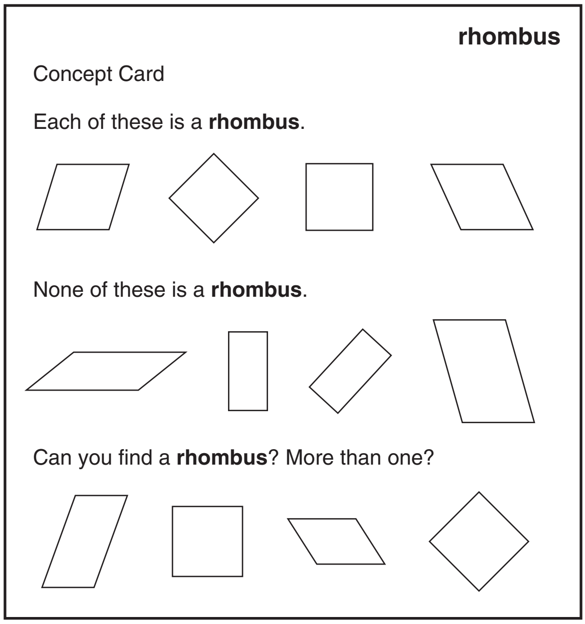


FIGURE 1.4 Concept card for rhombus.



or label, such as *rhombus*. Examples of a rhombus and non-examples of a rhombus are both shown on the card, and the student must decide what a rhombus is. Finally, the card provides an opportunity to test out his newly derived definition.

Students often learn erroneous concepts and processes similarly. They look for commonalities among their initial contacts with an idea or procedure. They form an abstraction with certain common characteristics, and their concept or algorithm is formed. The common attributes may be very specific, such as crossing out a digit, placing a digit in front of another, or finding the difference between two one-digit numbers (regardless of order). Failure to consider enough examples is one of the errors of inductive learning often cited by those who study thinking.

Because our students connect new information with what they already know, it is very important that we assess the preconceptions of our students. Prior knowledge is not always correct knowledge; misconceptions are common. Even when our students correctly observe particular characteristics that examples have in common, they may connect a pattern with a misconception and thereby learn an erroneous procedure.

When multiplication with fractions is introduced, students often have difficulty believing that correct answers make sense; throughout their previous experiences with factors and products, the product was always at least as great as the smaller factor. (Actually, the product is noticeably greater than either factor in most cases.) In the mind of these students, the concept *product* had come to include the idea of a greater number because this was common throughout most of their experiences with products.

From time to time, an erroneous procedure produces a correct answer. When it does, use of the error pattern is reinforced. For example, a student may decide that "rounding whole numbers to the nearest ten" means erasing the units digit and writing a zero. The student is correct about half of the time!

There are many reasons why students tend to learn patterns of error. It most certainly is not the intentional result of our instruction. Yet all too often, individual students do not have all the prerequisite understandings and skills they need when introduced to new ideas and procedures. When this happens, they may "grab at straws." A teacher who introduces paper-and-pencil procedures while a particular student still needs to work out problems with concrete aids encourages that student to try to memorize a complex sequence of mechanical acts, thereby prompting the student to adopt simplistic procedures that can be remembered. Because incorrect algorithms do not usually result in correct answers, it would appear that a student receives limited positive reinforcement for continued use of erroneous procedures. But students sometimes hold tenaciously to incorrect procedures, even during instruction that confronts their beliefs directly.

Students often invent similar rules when introduced to the sign for equals. For example, they may decide "The equals sign means 'the answer turns out to be.'"

Students who learn erroneous patterns *are* capable of learning. Typically, these students have what we might call a learn*ed* disability, not a learn*ing* disability. The rules that children construct are derived from their search for meaning; a sensible learning process is involved. This is true even for the erroneous rules they invent, though such rules may involve a distortion or a poor application. Sometimes students overgeneralize or overspecialize while learning.[15]

Overgeneralizing

Most of us are prone to overgeneralize on occasion; we "jump to a conclusion" before we have adequate data at hand. Examples of overgeneralizing abound in many areas of mathematics learning. Several interesting examples were observed by project staff at the University of Maryland during their study of misconceptions among secondary school students.[16] At the University of Pittsburgh, Mack studied the development of students' understanding of fractions during instruction, and she also observed students overgeneralizing.[17]

Consider the following examples of overgeneralizing.

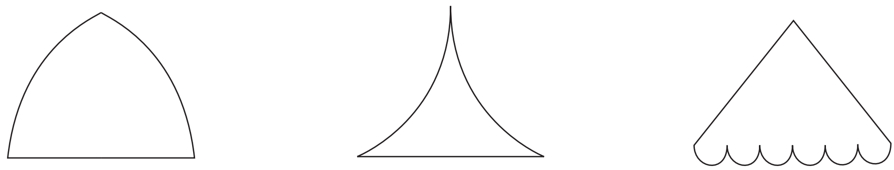
- What is a sum? Sometimes students decide that a sum is the number written on the right side of an equals sign.

$4 + 2 = 6$

Both are considered sums.

$6 - 2 = 4$

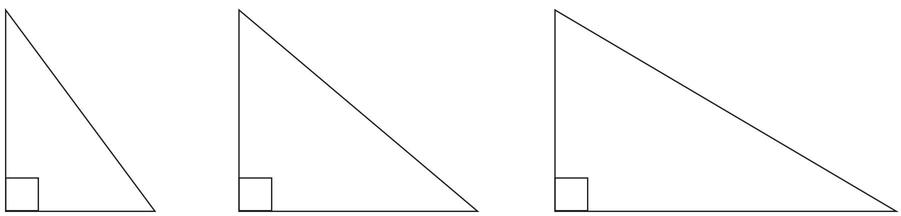
- Consider students who believe that all three of these figures are triangles.



Graeber reports an interesting speculation on this situation.

> **These students may be reasoning from a definition of triangle position. Extension of this definition to simple closed curves that are not polygons may lead to this error of including such shapes in the set of triangles.**[18]

- Sometimes students are exposed to right triangles like these:



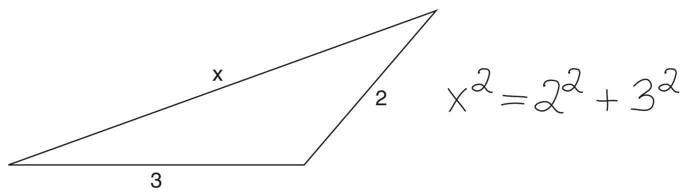
And they conclude that a right angle is oriented to the right as well as measuring 90 degrees.

a right angle . . . therefore . . . a left angle

- The student who believes that $2y$ means $20 + y$ may be overgeneralizing from expressions like $23 = 20 + 3$.
- Other students always use 10 for regrouping, even when computing with measurements.

$$\begin{array}{rrr} & 3 \text{ gal.} & 2 \text{ qt.} \\ - & 1 \text{ gal.} & 3 \text{ qt.} \\ \hline \end{array} \longrightarrow \begin{array}{rrr} & \overset{2}{\cancel{3}} \text{ gal.} & {}^{1}2 \text{ qt.} \\ - & 1 \text{ gal.} & 3 \text{ qt.} \\ \hline \end{array}$$

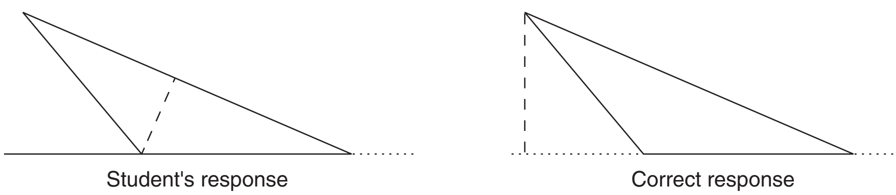
- Students sometimes think of the longest side of a triangle as a hypotenuse. They assume the Pythagorean Theorem applies even when the triangle is not a right triangle.[19]



Overspecializing

Other misconceptions and erroneous procedures are generated when a student overspecializes during the learning process. The resulting procedures are restricted inappropriately. For example, students know that in order to add or subtract fractions, the fractions must have like denominators. Sometimes students believe that multiplication and division of fractions require like denominators.

It is quite common for students to restrict their concept of altitude of a triangle to only that which can be contained within the triangle.



As we diagnose students who are experiencing difficulty, we need to be alert for both overgeneralization and overspecialization. We need to probe deeply as we examine written work—looking for misconceptions and erroneous procedures that form patterns across examples—and try to find out why specific procedures were learned.

Error Patterns in Computation

As our students learn concepts and computation procedures, many students—even students who invent their own algorithms—learn error *patterns*. Sometimes the words a teacher says are used (inappropriately) by a student when forming an error pattern. Chapters 2 through 8 include specific examples of error patterns—systematic procedures and applications of concepts students learn that often do not provide the correct answer.

Errors *can* be a positive thing. In many cultures, errors are regarded as an opportunity to reflect and learn. Furthermore, errors are often viewed as part of the "messiness" of doing mathematics.

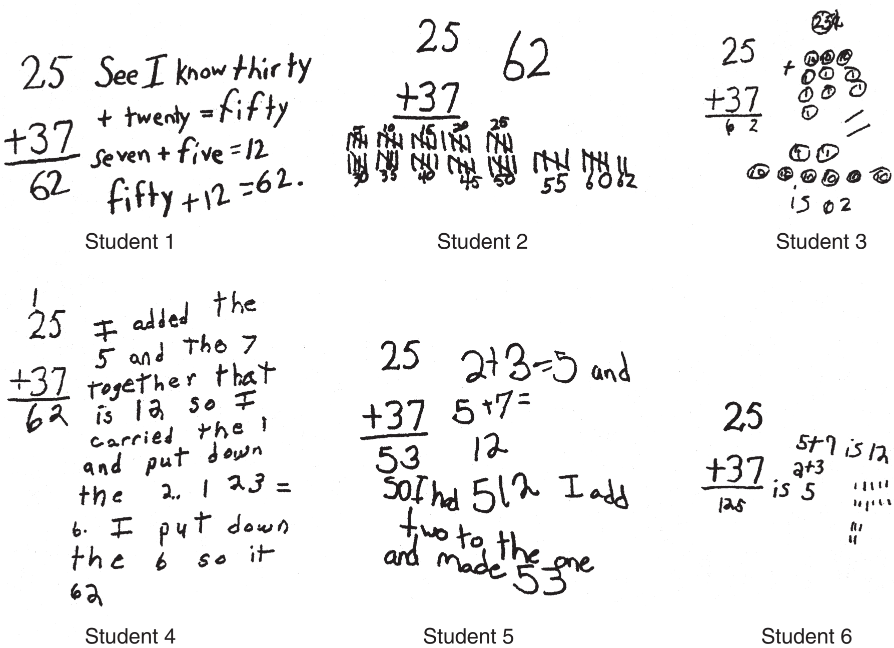
Rather than warning our students about errors to avoid, we can use errors as catalysts for learning by approaching errors as problem-solving situations. For example, a group of students can examine an erroneous procedure and use reasoning with concepts they already know to determine why the strategy that was used does not always produce a correct answer.[20]

Algorithms incorporating error patterns are sometimes called *buggy algorithms.* A buggy algorithm includes at least one erroneous step, and the procedure does not consistently accomplish the intended purpose.

As we teach computation procedures, we need to remember that our students are not necessarily learning what we think we are teaching; we need to keep our eyes and ears open to find out what our students are *actually* learning. We need to be alert for error patterns!

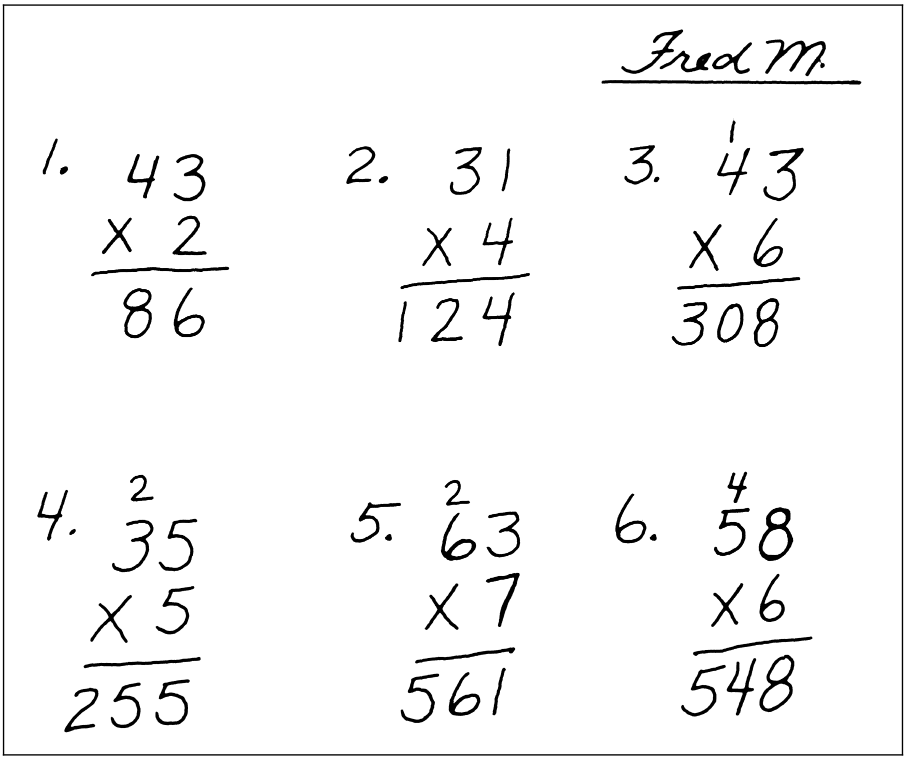
We can look for patterns as we examine student work, and note the different strategies for computing that our students develop. Of course, we observe that results are correct or incorrect; but we also need to look for evidence that indicates *how each student is thinking.* One way of getting at that thinking is to encourage students to show or describe how they obtained their answers, as can be seen for six students' solutions to 25 + 37 in Figure 1.5.

FIGURE 1.5 Six students' solutions to 25 + 37.



Source: Reproduced from Principles and Standards for School Mathematics (Reston, VA: National Council of Teachers of Mathematics, 2000) p. 85, by permission of National Council of Teachers of Mathematics.

FIGURE 1.6 Sample student paper.



Consider Fred's paper (Figure 1.6). If we merely determine how many answers are correct and how many are incorrect, we will not learn *why* his answers are not correct. Examine Fred's paper and note that when multiplication involves renaming, his answer is often incorrect. Look for a pattern among the incorrect responses; observe that he seems to be adding his "crutch" and then multiplying. This can be verified by studying other examples and briefly interviewing Fred.

Because we observed Fred's error pattern, instruction can be modified as needed. The algorithm may be reviewed as a written record of multiplying "one part at a time" (an application of distributing multiplication over addition), or a modification of the algorithm itself may be suggested so that the "crutch" is recorded with a small half-space digit written below the line. (See Error Pattern M-W-2 in Chapter 3.)

Rather than just scoring papers, we need to examine each student's paper diagnostically—looking for patterns, hypothesizing possible causes, and verifying our ideas. As we learn about each student, we will find that a student's paper is sometimes a problem or puzzle to be solved.

FURTHER REFLECTION

Consider the following questions:

1. Distinguish between *the meaning of addition* and *addition computation*.
2. For each of the four different methods of computation to be used for solving a problem, describe a social situation in which it makes sense to use that particular method.
3. How do the NCTM *process* standards support teaching of computation procedures in ways that help students make sense of those procedures?
4. How can you teach computation procedures so that students do *not* learn misconceptions and error patterns?

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   - Baroody, A. J., Feil, Y., & Johnson, A. R. (2007). An alternative reconceptualization of procedural and conceptual knowledge. *Journal for Research in Mathematics Education 38*(2), 115–131. A rejoiner by Star follows.
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