Chapter 1

Teaching Mathematics in the 21st Century

In this changing world, those who understand and can do mathematics will have significantly enhanced opportunities and options for shaping their futures. Mathematical competence opens doors to productive futures. A lack of mathematical competence keeps those doors closed. . . . All students should have the opportunity and the support necessary to learn significant mathematics with depth and understanding.

NCTM (2000, p. 50)

Someday soon you will find yourself in front of a class of students, or perhaps you are already teaching. What general ideas will guide the way you will teach mathematics? This book will help you become comfortable with the mathematics content of the pre-K–8 curriculum. You will also learn about research-based strategies for helping students come to know mathematics and be confident in their ability to do mathematics. These two things—your knowledge of mathematics and how students learn mathematics—are the most important tools you can acquire to be an effective teacher of mathematics. What you teach, however, is largely influenced by state and national standards, as well as local curriculum guides.

For more than two decades, mathematics education has been undergoing steady change. The impetus for this change, in both the content of school mathematics and the way mathematics is taught, can be traced to various sources, including knowledge gained from research. One significant factor in this change has been the professional leadership of the National Council of Teachers of Mathematics (NCTM), the world’s largest mathematics education organization, with more than 90,000 members (www.nctm.org). Another factor is the public or political pressure for change in mathematics education due largely to less-than-stellar U.S. student performance in national and international studies. The federal legislation commonly referred to as the No Child Left Behind Act (NCLB) presses for higher levels of achievement, more testing, and increased teacher accountability. Although all agree that we should have high expectations for students, there seems to be little consensus on what the best approach is to improve student learning. According to NCTM, “Learning mathematics is maximized when teachers focus on mathematical thinking and reasoning” (NCTM, 2009, n.d.).

As you prepare to help students learn mathematics, it is important to have some perspective on the forces that effect change in the mathematics classroom. This chapter addresses the leadership that NCTM provides for mathematics education as well as other important influences.

Ultimately, it is you, the teacher, who will shape mathematics for the students you teach. Your beliefs about what it means to know and do mathematics and about how students make sense of mathematics will affect how you approach instruction.

The National Standards-Based Movement

The momentum for reform in mathematics education began in the early 1980s in response to a “back to basics” movement that emphasized “reading, writing, and arithmetic.” As a result, problem solving became an important strand in the mathematics curriculum. The work of Jean Piaget and other developmental psychologists helped to focus research on how students can best learn mathematics.

This momentum came to a head in 1989, when NCTM published Curriculum and Evaluation Standards for School Mathematics and the standards movement or reform era in mathematics education began. It continues today. No other
document has ever had such an enormous effect on school mathematics or on any other area of the curriculum.

In 1991, NCTM published *Professional Standards for Teaching Mathematics*, which articulates a vision of teaching mathematics based on the expectation described in the *Curriculum and Evaluation Standards* that significant mathematics achievement is a vision for all students, not just a few. In 1995, NCTM added to the collection the *Assessment Standards for School Mathematics*, which focuses on the importance of integrating assessment with instruction and indicates the key role that assessment plays in implementing change (see Chapter 5).

In 2000, NCTM released *Principles and Standards for School Mathematics* as an update of its original standards document. Combined, these two standards documents have prompted a revolutionary reform movement in mathematics education, not just in the United States and Canada but throughout the world.

As these documents influenced state policy and teacher practice, ongoing debate continued about the U.S. curriculum. In particular, many argued that instead of hurrying through many topics every year, the curriculum needed to address content more deeply. Guidance was needed in deciding what mathematics content should be taught at each grade level. In 2006, NCTM released *Curriculum Focal Points*, a little publication with a big message—the mathematics taught at each grade level needs to focus, go into more depth, and explicitly show connections. The standards movement had gained significant momentum and engaged more than just the mathematics education community as business and political leaders became interested in a national vision for K–12 mathematics curriculum.

In 2010, the Council of Chief State School Officers (CCSSO) presented *Common Core State Standards*—grade-level specific standards that incorporated ideas from *Curriculum Focal Points* as well as international curriculum documents. A large majority of U.S. states adopted these as their standards. In less than 25 years, the standards movement transformed the country from having little to no national vision on what mathematics should be taught and when, to a widely shared vision of what students should know and be able to do at each grade level.

In the following sections, we discuss these more recent documents because their message is critical to your work as a teacher of mathematics.

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**Principles and Standards for School Mathematics**

*Principles and Standards for School Mathematics* (NCTM, 2000) provides guidance and direction for teachers and other leaders in pre-K–12 mathematics education.

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**The Six Principles**

One of the most important features of *Principles and Standards for School Mathematics* is the articulation of six principles fundamental to high-quality mathematics education:

- **Equity**
- **Curriculum**
- **Teaching**
- **Learning**
- **Assessment**
- **Technology**

According to *Principles and Standards*, these principles must be “deeply intertwined with school mathematics programs” (NCTM, 2000, p. 12). The principles make it clear that excellence in mathematics education involves much more than simply listing content objectives.

**The Equity Principle**

Excellence in mathematics education requires equity—high expectations and strong support for all students. (NCTM, 2000, p. 12)

The strong message of the Equity Principle is high expectations for all students. All students must have the opportunity and adequate support to learn mathematics “regardless of personal characteristics, backgrounds, or physical challenges” (p. 12). The significance of high expectations for all is interwoven throughout the document.

**The Curriculum Principle**

A curriculum is more than a collection of activities: it must be coherent, focused on important mathematics, and well articulated across the grades. (NCTM, 2000, p. 14)

Coherence speaks to the importance of building instruction around “big ideas”—both in the curriculum and in daily classroom instruction. Students must be helped to see that mathematics is an integrated whole, not a collection of isolated bits and pieces.

Mathematical ideas can be considered “important” if they help develop other ideas, link one idea to another, or serve to illustrate the discipline of mathematics as a human endeavor.

**The Teaching Principle**

Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well. (NCTM, 2000, p. 16)
What students learn about mathematics depends almost entirely on the experiences that teachers provide every day in the classroom. To provide high-quality mathematics education, teachers must (1) understand deeply the mathematics content they are teaching; (2) understand how students learn mathematics, including a keen awareness of the individual mathematical development of their own students and common misconceptions; and (3) select meaningful instructional tasks and generalizable strategies that will enhance learning. “Teachers’ actions are what encourage students to think, question, solve problems, and discuss their ideas, strategies, and solutions” (p. 18).

The Learning Principle

Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge. (NCTM, 2000, p. 20)

The learning principle is based on two fundamental ideas. First, learning mathematics with understanding is essential. Mathematics today requires not only computational skills but also the ability to think and reason mathematically to solve new problems and learn new ideas that students will face in the future.

Second, students can learn mathematics with understanding. Learning is enhanced in classrooms where students are required to evaluate their own ideas and those of others, are encouraged to make mathematical conjectures and test them, and are helped to develop their reasoning and sense-making skills.

The Assessment Principle

Assessment should support the learning of important mathematics and furnish useful information to both teachers and students . . . Assessment should not merely be done to students; rather, it should also be done for students, to guide and enhance their learning. (NCTM, 2000, p. 22)

Ongoing assessment highlights for students the most important mathematics concepts. Assessment that includes ongoing observation and student interaction encourages students to articulate and, thus, clarify their ideas. Feedback from daily assessment helps students establish goals and become more independent learners.

Assessment should be a major factor in making instructional decisions. By continuously gathering data about students’ understanding of concepts and growth in reasoning, teachers can better make the daily decisions that support student learning. For assessment to be effective, teachers must use a variety of assessment techniques, understand their mathematical goals deeply, and have a research-supported notion of students’ thinking or common misunderstandings of the mathematics that is being developed.
The process standards should not be regarded as separate content or strands in the mathematics curriculum. Rather, they direct the methods of doing all mathematics and, therefore, should be seen as integral components of all mathematics learning and teaching. To teach in a way that reflects these process standards is one of the best definitions of what it means to teach “according to the Standards.”

The Problem Solving standard describes problem solving as the vehicle through which students develop mathematical ideas. Learning and doing mathematics as you solve problems is probably the most significant message in the Standards documents.

The Reasoning and Proof standard emphasizes the logical thinking that helps us decide if and why our answers make sense. Students need to develop the habit of providing a rationale as an integral part of every answer. It is essential for students to learn the value of justifying ideas through logical argument.

The Communication standard points to the importance of being able to talk about, write about, describe, and explain mathematical ideas. Learning to communicate in mathematics fosters interaction and exploration of ideas in the classroom as students learn through active discussions of their thinking. No better way exists for wrestling with or cementing an idea than attempting to articulate it to others.

The Connections standard has two parts. First, it is important to connect within and among mathematical ideas. For example, fractional parts of a whole are connected to concepts of decimals and percents. Students need opportunities to see how mathematical concepts build on one another in a network of connected ideas.

Second, mathematics should be connected to the real world and to other disciplines. Students should see that mathematics plays a significant role in art, science, language arts, and social studies. This suggests that mathematics should frequently be integrated with other discipline areas and that applications of mathematics should be explored in real world contexts.

The Representation standard emphasizes the use of symbols, charts, graphs, manipulatives, and diagrams as powerful methods of expressing mathematical ideas and relationships. Symbolism in mathematics, along with visual aids such as charts and graphs, should be understood by students as ways of communicating mathematical ideas to others. Moving from one representation to another is an important way to add depth of understanding to a newly formed idea.

Members of NCTM have free online access to the Principles and Standards as well as the three previous standards documents. Nonmembers can sign up for 120 days of free access to the Principles and Standards at www.nctm.org.
Curriculum Focal Points: A Quest for Coherence

Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence (NCTM, 2006) pinpoints mathematical “targets” for each grade level that specify the big ideas for the most significant concepts and skills. At each grade, three essential areas (focal points) are described as the primary focus of that year’s instruction. The topics relating to that focus are organized to show the importance of a coherent curriculum rather than a curriculum with a list of isolated topics. The expectation is that three focal points along with integrated process skills and connecting experiences form the fundamental content of each grade. Besides focusing instruction, the document provides guidance to professionals about ways to refine and streamline curriculum in light of competing priorities.

Common Core State Standards

As noted earlier, the national dialogue on improving mathematics teaching and learning extends beyond mathematics educators. State policy makers and elected officials have also considered NCTM standards documents, international assessments, and research on the best way to prepare students to be “college and career ready.” The state governors (National Governors Association Center for Best Practices) and the Council of Chief State School Officers (CCSSO) collaborated with many other professional groups and entities to develop such benefits as shared expectations for K–12 students across states, a focused set of mathematics content standards and practices, and efficiency of material and assessment development (Porter, McMaken, Hwang, & Yang, 2011). As a result, they created the Common Core State Standards for Mathematics (which can be downloaded at www.corestandards.org). Like Curriculum Focal Points, this document articulates an overview of critical areas for each grade from kindergarten through 8 to provide a coherent curriculum built around big ideas. These larger groups of related standards are called domains, and there are eleven that relate to grades K–8 (see Figure 1.1).

At this time approximately 44 of the 50 states (and Washington, D.C., and the Virgin Islands) have adopted the Common Core State Standards. Notice that these standards are silent on preschool-aged students, so the use of the Curriculum Focal Points remains significant in making curricular decisions for this age group.

Mathematical Practice. The Common Core State Standards goes beyond specifying mathematics content to include Standards for Mathematical Practice. These are “processes and proficiencies’ with longstanding importance in mathematics education” (CCSSO, 2010, p. 6) that are founded on the five NCTM process standards and the components of mathematical proficiency identified by NRC in their important document Adding It Up (National Research Council, 2001). Teachers must develop these mathematical practices in all students (CCSSO, 2010, pp. 7–8) as described briefly in Table 1.2. (A more detailed description of the Standards for Mathematical Practice can be found in Appendix A.)

Learning Progressions. The Common Core State Standards were developed with strong consideration given to building coherence through the research on what is known about the development of students’ understanding of mathematics

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FIGURE 1.1 Common Core State Standards domains by grade level.
desired learning targets (Daro, Mosher, & Corcoran, 2011). Although these paths are not identical for all students, they can inform the order of instructional experiences that will support movement toward understanding and application of mathematics concepts. Go to http://math.arizona.edu/~ime/progressions to find progressions for the domains in the Common Core State Standards.

Assessments. New summative assessments are being developed that will be aligned to the Common Core State Standards. The assessments will focus on both the grade-level content standards and the standards for mathematical practice. This process would eliminate the need for each

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**TABLE 1.2**

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<th>THE STANDARDS FOR MATHEMATICAL PRACTICE FROM THE COMMON CORE STATE STANDARDS</th>
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<td>K–8 Students Should Be Able To:</td>
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| Make sense of problems and persevere in solving them | • Explain the meaning of a problem  
  • Describe possible approaches to a solution  
  • Consider similar problems to gain insights  
  • Use concrete objects or illustrations to think about and solve problems  
  • Monitor and evaluate their progress and change strategy if needed  
  • Check their answers using a different method |
| Reason abstractly and quantitatively | • Explain the relationship between quantities in problem situations  
  • Represent situations using symbols (e.g., writing expressions or equations)  
  • Create representations that fit the problem  
  • Use flexibly the different properties of operations and objects |
| Construct viable arguments and critique the reasoning of others | • Understand and use assumptions, definitions, and previous results to explain or justify solutions  
  • Make conjectures by building a logical set of statements  
  • Analyze situations and use counterexamples  
  • Justify conclusions in a way that is understandable to teachers and peers  
  • Compare two possible arguments for strengths and weaknesses |
| Model with mathematics | • Apply mathematics to solve problems in everyday life  
  • Make assumptions and approximations to simplify a problem  
  • Identify important quantities and use tools to map their relationships  
  • Reflect on the reasonableness of their answer based on the context of the problem |
| Use appropriate tools strategically | • Consider a variety of tools and choose the appropriate tool (e.g., manipulative, ruler, technology) to support their problem solving  
  • Use estimation to detect possible errors  
  • Use technology to help visualize, explore, and compare information |
| Attend to precision | • Communicate precisely using clear definitions and appropriate mathematics language  
  • State the meanings of symbols  
  • Specify appropriate units of measure and labels of axes  
  • Use a degree of precision appropriate for the problem context |
| Look for and make use of structure | • Explain mathematical patterns or structures  
  • Shift perspective and see things as single objects or as composed of several objects  
  • Explain why and when properties of operations are true in a context |
| Look for and express regularity in repeated reasoning | • Notice if calculations are repeated and use information to solve problems  
  • Use and justify the use of general methods or shortcuts  
  • Self-assess to see whether a strategy makes sense as they work, checking for reasonableness prior to getting the answer |

Source: Adapted from Council of Chief State School Officers. (2010). *Common Core State Standards*. Copyright © 2010 National Governors Association Center for Best Practices and Council of Chief State School Officers. All rights reserved.

over time (Cobb & Jackson, 2011). The resulting selections of topics at particular grades reflects not only rigorous mathematics but also what is known from current research and practice about learning progressions—sometimes referred to as learning trajectories (Confrey, Maloney, & Nguyen, 2011; Daro, Mosher, & Corcoran, 2011; Sarama & Clements, 2009) or teaching-learning paths (Cross, Woods, & Schweingruber, 2009). It is these learning progressions that can help teachers know what came before as well as what to expect next as students reach “key waypoints” (Corcoran, Mosher, & Rogat, 2009) on the road to learning mathematics concepts. These progressions identify the interim goals students should reach on the path to desired learning targets (Daro, Mosher, & Corcoran, 2011). Although these paths are not identical for all students, they can inform the order of instructional experiences that will support movement toward understanding and application of mathematics concepts. Go to http://math.arizona.edu/~ime/progressions to find progressions for the domains in the Common Core State Standards.
In addition to curriculum-related standards, NCTM has developed related standards documents about teaching. Professional Standards for Teaching Mathematics (1991) and its companion document, Mathematics Teaching Today (2007a), use detailed classroom stories (vignettes) of real teachers to illustrate the careful, reflective work that is required of effective teachers of mathematics. Mathematics Teaching Today and its predecessor are excellent resources to help you envision your role as a teacher in creating a classroom that supports teaching through problem solving. As you read the chapters in this book, you will note that the following seven standards are developed in ways that will support your growth as a teacher of mathematics. (See Appendix B for detailed descriptions of these standards.)

1. Knowledge of Mathematics and General Pedagogy
2. Knowledge of Student Mathematical Learning
3. Worthwhile Mathematical Tasks
4. Learning Environment
5. Discourse
6. Reflection on Student Learning
7. Reflection on Teaching Practice

Mathematics Teaching Today lists six major components of the mathematics classroom that are necessary to allow students to develop mathematical understanding:

- Creating an environment that offers all students an equal opportunity to learn
- Focusing on a balance of conceptual understanding and procedural fluency
- Ensuring active student engagement in the NCTM process standards (problem solving, reasoning, communication, connections, and representation)
- Using technology to enhance understanding
- Incorporating multiple assessments aligned with instructional goals and mathematical practices
- Helping students recognize the power of sound reasoning and mathematical integrity (NCTM, 2007a).

**Trends in International Mathematics and Science Study (TIMSS).** In the mid-1990s, 41 nations participated in the Third International Mathematics and Science Study, the largest study of mathematics and science education ever conducted. Data were gathered in grades 4, 8, and 12 from 500,000 students as well as from teachers. The most widely reported results revealed that U.S. students performed above the international average of the TIMSS countries at the fourth grade, below the average at the eighth grade, and significantly below average at the twelfth grade (U.S. Department of Education, 1997a).

TIMSS studies were repeated in 1999 (38 countries), 2003 (46 countries), and again in 2007 (63 countries). (See http://nces.ed.gov/timss for details.) The 2007 TIMSS found that U.S. fourth and eighth graders were above the international average, but were significantly outperformed by eight countries or parts of countries (Hong Kong, Singapore, Chinese Taipei, Japan, Kazakhstan, Russian Federation, England, and Latvia) at the fourth-grade level and by five countries (Chinese Taipei, Korea, Singapore, Hong Kong, and Japan) at the eighth-grade level. Only 15 percent of U.S. fourth graders and 10 percent of U.S. eighth graders performed above the advanced international benchmark. This is in stark contrast with Singapore at 44 percent at the fourth grade and 32 percent at the eighth grade. The impressive performance by Singapore has led some educators to talk about “Singapore mathematics” as a methodology to be emulated.

A report on the original TIMSS curriculum analysis labeled the U.S. mathematics curriculum “a mile wide and an inch deep” (Schmidt, McKnight, & Raizen, 1996, p. 62), meaning it was found to be unfocused, pursuing many more topics than other countries while engaging in a great deal of repetition. They found the U.S. curriculum attempted to do everything and, as a consequence, rarely provided depth of study, making reteaching all too common.

One of the most interesting components of the study was the videotaping of eighth-grade classrooms in the United States, Australia, and five of the highest-achieving countries. The results indicate that teaching is a cultural activity and, despite similarities, the differences in the ways countries taught mathematics were often striking. In all countries, problems or tasks were frequently used to begin the lesson. However, as a lesson progressed, the way these problems were handled in the United States was in stark contrast to high-achieving countries.

Analysis revealed that although the world is for all purposes unrecognizable from what it was 100 years ago, the U.S. approach to teaching mathematics during the same time frame was essentially unchanged (Stigler & Hiebert, 2009). They found the U.S. teacher begins with a review of previous materials or homework and then demonstrates a problem. Students practice similar problems at their desks, the teacher checks the seatwork, and then assigns problems for either the remainder of the class session or homework. (Sound familiar?) In more than 99.5 percent of the U.S. lessons, the teacher reverts to showing students how to solve the problems. In not one of the 81 U.S. lessons was any high-level mathematics content observed; in contrast, 30 to 40 percent of lessons in Germany and Japan contained high-level mathematics content. Although all teachers knew the research team was coming to videotape, 89 percent of the U.S. lessons consisted exclusively of low-level content. Other countries incorporated a variety of methods, but they frequently used a problem-solving approach with an emphasis on conceptual understanding and students engaged in problem solving (Hiebert et al., 2003). Teaching in the high-achieving countries more closely resembles the recommendations of the NCTM standards than does the teaching in the United States.

**Curriculum**

As described in the beginning of this chapter, curriculum documents (standards) have a significant influence on what is taught, and even how it is taught. In addition, the textbook is a very influential factor in determining the what, when, and how of actual teaching. What is becoming increasingly complicated is how teachers and school systems attempt to align existing textbooks or other curriculum materials with the Common Core State Standards, Curriculum Focal Points, or other key documents.

Textbooks greatly influence teaching practice. A teacher using one textbook may be more likely to cover many topics, spend one day on each topic, use a teacher-directed instructional approach, and focus on procedures. Using a different textbook (that is more standards-based), a teacher may devote more time to a concept, teach it more deeply, and use a student-centered approach. Writing, speaking, working in groups, and problem solving are more likely to be commonplace components in current curriculum offerings. The selection of curriculum materials is an important endeavor.

In Section II of this book you will find features describing activities from two standards-based (problem-solving oriented) curriculum programs: *Investigations in Number, Data, and Space (Grades K–5)* and *Connected Mathematics Project (Grades 6–8)*. These features are included to offer you some insight into how a textbook can support your implementation of the standards (both the content and the processes/practices).

**A Changing World Economy**

In his book *The World Is Flat* (2007), Thomas Friedman discusses the need for people to have skills that are lasting and will survive the ever-changing landscape of available
jobs. These are what he calls “the untouchables”—the individuals who outlast all the ups and downs of the economy. He suggests people who fit into several broad categories that he defines will not be challenged by a shifting job market. One of his safety-ensuring categories is “math lovers.” Friedman points out that in a world that is digitized and surrounded by algorithms, math lovers will always have career opportunities and options.

Now it becomes the job of the teacher to develop this passion in students. As Lynn Arthur Steen, a well-known mathematician and educator, states, “As information becomes ever more quantitative and as society relies increasingly on computers and the data they produce, an innumerate citizen today is as vulnerable as the illiterate peasant of Gutenberg’s time” (1997, p. xv).

The changing world influences what should be taught in pre-K–8 mathematics classrooms. As we prepare pre-K–8 students for jobs that possibly do not currently exist, we do know that there are few jobs for people where they just do simple computation. We can predict that there will be work that requires interpreting complex data, designing algorithms to make predictions, and using the ability to approach new problems in a variety of ways.

An Invitation to Learn and Grow

The mathematics education described in the NCTM Principles and Standards and new Common Core State Standards may not be the same as the mathematics and the mathematics teaching you experienced in grades K through 8. Along the way, you may have had excellent teachers of mathematics who reflect the current reform spirit. Examples of good standards-based curriculum have been around since the early 1990s. But for the most part, after more than two decades the goals of the reform movement have yet to be realized in many of the school districts in North America.

As a practicing or prospective teacher facing the challenge of teaching standards-based mathematics from a problem-solving approach, this book may require you to confront some of your personal beliefs—about what it means to do mathematics, how one goes about learning mathematics, how to teach mathematics through reasoning and sense making, and what it means to assess mathematics so that it leads to targeted instructional change.

As part of this personal assessment, you should understand that mathematics is unfortunately seen as the subject that people love to hate. At parties or even at parent–teacher conferences, other adults will respond to the fact that you are a teacher of mathematics with comments such as “I could never do math,” or “I can’t even balance my checking account.” Instead of just dismissing these disclosures, consider what action you can take. Would people confide that they don’t read and hadn’t read a book in years? That is not likely. Families’ and teachers’ attitudes toward mathematics may enhance or detract from students’ ability to do math. It is important for you and for students’ families to know that mathematics ability is not inherited—anyone can learn mathematics. Moreover, learning mathematics is an essential life skill. You need to find ways of countering these statements, especially if they are stated in the presence of students, pointing out the importance of the topic and the fact that all people have the capacity to learn mathematics. Only in that way can the long-standing sequence that passes this apprehension from family member to child (or in rare cases teacher to student) be broken. There is much joy to be had in solving mathematical problems, and you need to model this and nurture that passion in your students.

Students and adults alike need to think of themselves as mathematicians, in the same way as many think of themselves as readers. As all people interact with our increasingly mathematical and technological world, they need to construct, modify, or integrate new information in many forms. Solving novel problems and approaching circumstances with a mathematical perspective should come as naturally as reading new materials to comprehend facts, insights, or news. Consider how important this is to interpreting and successfully surviving in our economy. Thinking and talking about mathematics instead of focusing on the “one right answer” is a strategy that will serve us well in becoming a society where all citizens are confident that they can do math.

Becoming a Teacher of Mathematics

This book and this course of study are critical to your professional teaching career. The mathematics education course you are taking now or the professional development you are experiencing will be the foundation for much of the mathematics instruction you do in your classroom for the next decade. The authors of this book take that seriously, as we know you do. Therefore, this section lists and describes the characteristics, habits of thought, skills, and dispositions you will need to succeed as a teacher of mathematics.

Knowledge of Mathematics. You will need to have a profound, flexible, and adaptive knowledge of mathematics content (Ma, 1999). This statement is not intended to scare you if you feel that mathematics is not your strong suit, but it is meant to help you prepare for a serious semester of learning about mathematics and how to teach it. The “school effects” for mathematics are great, meaning that unlike other subject areas, where students have frequent interactions with their family or others outside of school on topics such as reading books, exploring nature, or discussing current events, in the area of mathematics what we do in school is often “it” for many students. This adds to the earnestness of your responsibility, because a student’s learning for the year in mathematics will likely come only from you.
If you are not sure of a fractional concept or other aspect of mathematics content knowledge, now is the time to make changes in your depth and flexibility of understanding to best prepare for your role as an instructional leader. This book and your professor will help you in that process.

**Persistence.** You need the ability to stave off frustration and demonstrate persistence. This is the very skill that your students must have to conduct mathematical investigations. As you move through this book and work the problems yourself, you will learn methods and strategies that will help you anticipate the barriers to student learning and identify strategies to get them past these stumbling blocks. It is likely that what works for you as a learner will work for your students. As you experience the material in this book, if you ponder, struggle, talk about your thinking, and reflect on how it all fits or doesn’t fit your prior knowledge, then you enhance your repertoire as a teacher. Remember you need to demonstrate these characteristics so your students can model them. Creating opportunities for your students to struggle is part of learning (Stigler & Hiebert, 2009).

**Positive Attitude.** Arm yourself with a positive attitude toward the subject of mathematics. Research shows that teachers with positive attitudes teach math in more successful ways that result in their students liking math more (Karp, 1991). If in your heart you say, “I never liked math,” that will be evident in your instruction. The good news is that research shows that changing attitudes toward mathematics is relatively easy (Tobias, 1995) and that attitude changes are long-lasting (Dweck, 2006). Through expanding your knowledge of the subject and trying new ways to approach problems, you can learn to enjoy mathematical activities. Not only can you acquire a positive attitude toward mathematics, it is essential that you do.

**Readiness for Change.** Demonstrate a readiness for change, even for change so radical that it may cause disequilibrium. You may find that what is familiar will become unfamiliar and, conversely, what is unfamiliar will become familiar. For example, you may have always referred to “reducing fractions” as the process of changing $\frac{1}{2}$ to $\frac{1}{4}$, but this phrase is not appropriate because it is misleading—the fractions are not getting smaller. Such terminology can lead to mistaken connections. (“Did the reduced fraction go on a diet?”). A careful look will point out that reducing is not the term to use; rather, you are writing an equivalent fraction that is simplified. Even though you have used the term reducing for years, you need to become familiar with more precise language such as “simplifying fractions.”

On the other hand, what is unfamiliar will become more comfortable. It may feel uncomfortable for you to be asking students, “Did anyone solve it differently?” especially if you are worried that you won’t understand their approach. Yet this is essential to effective teaching. As you bravely use this strategy, it will become comfortable (and you will learn new things!).

Another potentially difficult change is toward an emphasis on concepts as well as procedures. What happens in a procedure-focused classroom when a student doesn’t understand division of fractions? A teacher with only procedural knowledge is often left to repeat the procedure louder and slower. “Just change the division sign to multiplication, flip over the second fraction, and multiply.” We know this approach doesn’t work well, so let’s consider an example using $\frac{3}{2} \div \frac{1}{2} = \square$. You might relate this division problem to a whole number division problem such as $25 \div 5 = \square$. A corresponding story problem might be, “How many orders of 5 pizzas are there in a group of 25 pizzas?” Then ask students to put words around the fraction division problem, such as “You plan to serve each guest $\frac{1}{2}$ a pizza. If you have $\frac{3}{2}$ pizzas, how many guests can you serve?” Yes, there are seven halves in $\frac{3}{2}$ and therefore 7 guests can be served. Are you surprised that you can do this problem in your head?

To respond to students’ challenges, uncertainties, and frustrations you may need to unlearn and relearn mathematical concepts, developing comprehensive understanding and substantial representations along the way. Supporting your mathematics content knowledge on solid, well-supported terrain is your best hope of making a lasting difference—so be ready for change. What you already understand will provide you with many “Aha!” moments as you read this book and connect new information to your current mathematics knowledge.

**Reflective Disposition.** Make time to be self-conscious and reflective. As Steve Leinwand wrote, “If you don’t feel inadequate, you’re probably not doing the job” (2007, p. 583). No matter whether you are a preservice teacher or an experienced teacher, there is more to learn about the content and methodology of teaching mathematics. The ability to examine oneself for areas that need improvement or to reflect on successes and challenges is critical for growth and development. The best teachers are always trying to improve their practice through the latest article, the newest book, the most recent conference, or by signing up for the next series of professional development opportunities. These teachers don’t say, “Oh, that’s what I am already doing”; instead, they identify and celebrate one small tidbit that adds to their repertoire. The best teachers never finish learning all that they need to know, they never exhaust the number of new mental connections that they make, and, as a result, they never see teaching as stale or stagnant. An ancient Chinese proverb states, “The best time to plant a tree is twenty years ago; the second best time is today.” So, as John Van de Walle said with every new edition, “Enjoy the journey!”
RESOURCES for Chapter 1

RECOMMENDED READINGS

Articles
Hoffman, L., & Brahier, D. (2008). Improving the planning and teaching of mathematics by reflecting on research. Mathematics Teaching in the Middle School, 13(7), 412–417. This article addresses how a teacher’s philosophy and beliefs influence his or her mathematics instruction. Using TIMSS and NAEP studies as a foundation, the authors talk about posing higher-level problems, asking thought-provoking questions, facing students’ frustration, and using mistakes to enhance understanding of concepts. They pose a set of reflective questions that are good for self-assessment or discussions with peers.

Books
Lambdin, D., & Lester, F. K., Jr. (2010). Teaching and learning mathematics: Translating research for elementary school teachers. Reston, VA: NCTM. Using the most current research on the teaching and learning of mathematics, this book translates research into meaningful chapters for classroom teachers. Built around major questions on a variety of topics, the authors highlight the importance of research in helping teachers be reflective and to assist in the day-to-day judgments teachers make as they support all learners.


ONLINE RESOURCES

Dare to Compare (NCES Kids’ Zone)
See how your students perform compared to peers from around the world on items used on past administrations of the grades 4 and 8 NAEP and grades 4, 8, and 12 TIMSS.

Illuminations
http://illuminations.nctm.org
A companion website to NCTM provides lessons, interactive applets, dynamic paper, and links to websites for learning and teaching mathematics.

Illustrative Mathematics Project
http://illustrativemathematics.org
This site provides tools and support for the Common Core State Standards. It includes multiple ways to look at the standards across grades and domains as well as provides task and problems that will illustrate individual standards.

National Council of Teachers of Mathematics
www.nctm.org
Here you can discover everything about NCTM and its resources to support your work. Also find an overview of several standards-based documents, position statements, research-based clips and briefs, free access to interactive digital lessons, professional development resources, membership and conference information, online publications store, links to related sites, and much more.

Progressions Documents for the Common Core Math Standards
http://math.arizona.edu/~ime/progressions
This site provides the learning progressions based on mathematical structure and students’ cognitive development at given grades across the domains in the Common Core State Standards.

DISCUSSION QUESTION for Chapter 1

1. In NCTM’s Mathematics Teaching Today are six components of the mathematics classroom (p. 7) that are suggested as necessary to allow students to develop mathematical understanding. Examine these shifts and discuss which aspects of each shift seem the most significant to you including why these are the most significant to you and/or your students.

PROFESSIONAL DEVELOPMENT ACTIVITIES

1. Ask teachers to examine a textbook at any grade level of their choice. If possible, use a teacher’s edition. Ask them to page through any chapter and look for signs of the five NCTM process standards or the eight CCSSO Standards for Mathematical Practice. Ask them to what extent students who are being taught from that textbook are likely to be doing and learning mathematics in ways described by those processes.
or practices. What would you have to do to supplement the general approach of that text to meet these goals?

2. Examine the Standards for Teaching Mathematics (Appendix B). As teachers read the list, have them code each in the following manner using these three symbols. (You can use a gallery walk approach for this too.)

- the practices that excite you
- the practices that you wonder about or question
- the practices that are reflective of your current practice (individually, grade level team, school)

Discuss in small groups. Ask teachers to set one to two goals for their mathematics instructional practice informed by this coding and discussion.

3. Read the article “Improving the Planning and Teaching of Mathematics by Reflecting on Research” listed in the Recommended Readings section. Begin by discussing the major findings from the TIMSS study. Then use the set of reflective questions in the article to prompt analysis of teachers’ philosophies and actions in their classrooms.

Among the many tools on PDToolkit, the following tools provide support in teaching mathematics in light of the NCTM standards:

- Video Using Manipulatives as Models as a basis to discuss the NCTM standards in action
- NCTM Process Standards classroom observation tool
- Teacher reflection tools with the indicated foci:
  - Student Understanding
  - Reflecting on Teaching and Learning
  - Reactions to Learning with Technology
  - Professional Growth

CoACH/TEACHER LEADER

CONSIDERATIONS

- Informed by the goal setting from Professional Development Activity 2, the coach should make a plan with the teacher about how to best support her/him in the process of reaching her/his goal(s) and should regularly check in regarding progress toward the goal(s), celebrating progress along the way.
Chapter 2
Exploring What It Means to Know and Do Mathematics

No matter how lucidly and patiently teachers explain to their students, they cannot understand for their students.
Schifter and Fosnot (1993, p. 9)

What does it mean to know a mathematics topic? Take division of fractions, for example. If you know this topic well, what do you know? The answer is more than being able to do a procedure (e.g., invert the second fraction and multiply). Knowing division of fractions means that you can think of examples or situations, use alternative strategies to solve problems, estimate an answer, draw a diagram to show what happens when a number is divided by a fraction, and describe in general what it means.

This chapter is about the learning theory of teaching developmentally and the knowledge necessary for students to learn mathematics with understanding. You might consider this chapter the what, why, and how of teaching mathematics. The how is addressed first—how should mathematics be experienced by a learner? Second, why should mathematics look this way? And, finally, what does it mean to understand mathematics?

What Does It Mean to Do Mathematics?
Mathematics is more than completing sets of exercises or mimicking processes the teacher explains. Doing mathematics means generating strategies for solving problems, applying those approaches, seeing if they lead to solutions, and checking to see whether your answers make sense. Doing mathematics in classrooms should closely model the act of doing mathematics in the real world.

Mathematics Is the Science of Pattern and Order
This heading is a wonderfully simple description of mathematics, found in the thought-provoking publication Everybody Counts (Mathematical Sciences Education Board, 1989). This emphasis challenges the popular view of mathematics as a discipline dominated by computation. Science is a process of figuring out or making sense, and mathematics is the science of concepts and processes that have a pattern of regularity and logical order. Finding and exploring this regularity or order, and then making sense of it, is what doing mathematics is all about.

Even the youngest schoolchildren can and should be involved in the science of pattern and order. Have you ever noticed that these combinations all have the same sum?

\[
\begin{align*}
6 + 7 &= 13 \\
5 + 8 &= 13 \\
4 + 9 &= 13 
\end{align*}
\]

Do you see a pattern? What are the relationships between these examples? In multiplication, have you ever wondered why an odd number times an odd number always generates an odd answer, an even number times an even number is always an even number, and an even number times an odd number can be either an even or an odd number? Why is this true?

Patterns are central to algebra, too. Imagine sending a toy car down a ramp. Does the height of the ramp determine how far the car will roll? Through exploring different ramp heights and measuring the distance the toy cars travel, you can see whether there is a pattern, which leads to a general rule—a function—to describe the relationship between ramp height and distance traveled by the car.

Engaging in the science of pattern and order—doing mathematics—takes time and effort (for teachers as they...
Beyond the required response, and (3) conjecturing or to previous material, (2) responding with information to take the math ideas to the next level by (1) connecting “doers” of mathematics. In other words the students began mathematics, researchers found that students became room where the teacher used this approach to teaching ideas that are involved. In observing a third-grade class-classes will be actively thinking about the mathematical thinking and encompass “making sense” and “figuring out.” Children engaged in these actions in mathematics doing mathematics than playing scales on the piano is making music.

**PAUSE and REFLECT**

Envision for a moment an elementary or middle school mathematics class where students are doing mathematics as “a study of patterns.” What do you see as you observe this class? Think of three ideas, and then read about the classroom environment.

### A Classroom Environment for Doing Mathematics

Doing mathematics begins with posing worthwhile tasks and then creating an environment where students take risks and share and defend mathematical ideas. Students are actively engaged in solving problems, and teachers are posing questions that encourage students to make connections and understand the mathematics they are exploring.

**The Language of Doing Mathematics.** Children in traditional mathematics classes often describe mathematics as imitating what the teacher shows them. Instructions to students given by teachers or in textbooks ask students to listen, copy, memorize, drill, and compute. These are lower-level thinking activities and do not adequately prepare students for the real act of doing mathematics. In contrast, the following verbs engage students in doing mathematics:

- compare
- conjecture
- construct
- describe
- develop
- explain
- explore
- formulate
- investigate
- justify
- predict
- represent
- solve
- use
- verify

These verbs lead to opportunities for higher-level thinking and encompass “making sense” and “figuring out.” Children engaged in these actions in mathematics classes will be actively thinking about the mathematical ideas that are involved. In observing a third-grade classroom where the teacher used this approach to teaching mathematics, researchers found that students became “doers” of mathematics. In other words the students began to take the math ideas to the next level by (1) connecting to previous material, (2) responding with information beyond the required response, and (3) conjecturing or predicting (Fillingim & Barlow, 2010). When this happens on a daily basis, students are getting an empowering message: “You are capable of making sense of this—you are capable of doing mathematics!”

**The Classroom Environment for Doing Mathematics.** Classrooms where students are making sense of mathematics do not happen by accident—they happen because the teacher establishes practices and expectations that encourage risk taking, reasoning, sharing, and so on. The list below provides expectations that are often cited as ones that support students in doing mathematics (Clarke & Clarke, 2004, CCSSO, 2010, Hiebert et al., 1997, NCTM, 2007).

1. **Perseverance, effort, and concentration are important in learning mathematics.** Engaging in productive struggle is important in learning! The more a student stays with a problem, the more likely they are to get it right. Getting a tough problem right leads to a stronger sense of accomplishment than getting a quick, easy problem correct.
2. **Students share their ideas.** Everyone’s ideas are important, and hearing different ideas helps students to become strategic in selecting good strategies.
3. **Students listen to each other.** All students have something to contribute and these ideas should be considered and evaluated for whether they will work in that situation.
4. **Errors or strategies that didn’t work are opportunities for learning.** Mistakes are opportunities for learning—why did that approach not work? Could it be adapted and work or is a completely different approach needed? Doing mathematics involves monitoring and reflecting on the process—catching and adjusting errors along the way.
5. **Students look for and discuss connections.** Students should see connections between different strategies to solve a particular problem, as well as connections to other mathematics concepts and to real contexts and situations. When students look for and discuss connections, they see mathematics as worthwhile and important, rather than an isolated collection of facts.

Notice who is doing the thinking, the talking, and the mathematics—the students. Mathematics requires effort, and it is important that students, families, and the community acknowledge and honor the fact that effort is what leads to learning in mathematics (National Mathematics Advisory Panel, 2008). In fact, a review of research on what connects mathematics teaching practice to student learning found that two things result in conceptual understanding: making mathematics relationships explicit and engaging students in productive struggle (Bay-Williams, 2010; Hiebert & Grouws, 2007).

The teacher’s role is making mathematical relationships explicit is to be sure that students are making the connections that are implied in a task. For example, asking students to relate today’s topic to one they investigated last week, or by asking “How is Lisa’s strategy like Marco’s strategy?” when the two students have picked different ways to...
solve a problem, are ways to be “explicit” about mathematical relationships. The focus is on students’ applying their prior knowledge, testing ideas, making connections and comparisons, and making conjectures.

Have you ever just wanted to think through something yourself, without being interrupted or told how to do it? Yet, how often in mathematics class does this happen? As soon as a student pauses in solving a problem the teacher steps in to show or explain. While this may initially help the student reach the answer, it does not help the student learn mathematics—engaging in productive struggle is what helps students learn mathematics. Notice the importance of both words in “productive struggle.” Students must have the tools and prior knowledge to solve a problem, and not be given a problem that is out of reach, or they will struggle without being productive; yet students should not be given tasks that are straightforward and easy, or they will not be struggling with mathematical ideas. When students (even very young students) know that struggle is expected as part of the process of doing mathematics, they embrace the struggle and feel success when they reach a solution (Carter, 2008).


d is an Invitation to Do Mathematics

The purpose of this section is to provide you with opportunities to engage in the science of pattern and order—to do some mathematics. If possible, find one or two peers to work with you so that you can experience sharing and exchanging ideas. For each problem posed, allow yourself to try to (1) make connections within the mathematics (i.e., make mathematical relationships explicit) and (2) engage in productive struggle.

We will explore four different problems. None requires mathematics beyond elementary school mathematics—not even algebra. But the problems do require higher-level thinking and reasoning. Try out your ideas! Have fun!

Problems

1. Start and Jump Numbers: Searching for Patterns
You will need to make a list of numbers that begin with a “start number” and increase by a fixed amount we will call the “jump number.” First try 3 as the start number and 5 as the jump number. Write the start number at the top of your list, then 8, 13, and so on, “jumping” by 5 each time until your list extends to about 130.

Examine this list of numbers and record the patterns you see. Share your ideas with the group, and write down every pattern you agree really is a pattern.

A Few Ideas. Here are some questions to guide your pattern search:

- Do you see at least one alternating pattern?
- Have you looked at odd and even numbers?
- What can you say about the numbers in the tens place?
- Have you tried doing any adding of numbers? Numbers in the list? Digits in the numbers?
- Do the patterns change when the numbers are greater than 100?

Don’t forget to think about what happens to your patterns after the numbers are more than 100. How are you thinking about 113? One way is as 1 hundred, 1 ten, and 3 ones. But, of course, it could also be “eleventy-three,” where the tens digit has gone from 9 to 10 to 11. How do these different perspectives affect your patterns? What would happen after 999?

When you added the digits in the numbers, the sums are 3, 8, 4, 9, 5, 10, 6, 11, 7, 12, 8, and so on. Did you look at every other number in this string? And what is the sum of the digits for 113? Is it 5 or is it 14? (There is no “right” answer here. But it is interesting to consider different possibilities.)

Next Steps. Sometimes when you have discovered some patterns in mathematics, it is a good idea to make some changes and see how the changes affect the patterns. What changes might you make in this problem?

Your changes may be even more interesting than the following suggestions. But here are some ideas:

- Change the start number but keep the jump number equal to 5. What is the same and what is different?
- Keep the same start number and explore with different jump numbers.
- What patterns do different jump numbers make? For example, when the jump number was 5, the ones-digit pattern repeated every two numbers—it had a “pattern length” of 2. But when the jump number is 3, the length of the ones-digit pattern is 10! Do other jump numbers create different pattern lengths?
of paper in 4 hours. The new machine could shred the same truckload in only 2 hours. How long will it take to shred a truckload of paper if Ron runs both shredders at the same time?

For a jump number of 3, how does the ones-digit pattern relate to the circle of numbers in Figure 2.1? Are there similar circles of numbers for other jump numbers?

Using the circle of numbers for 3, find the pattern for jumps of multiples of 3, that is, jumps of 6, 9, or 12.

Using Technology. Calculators facilitate exploration of this problem. Using the calculator makes the list generation accessible for young children who can’t skip count yet, and it opens the door for students to work with bigger jump numbers, such as 25 or 36. Most simple calculators have an automatic constant feature that will add the same number successively. For example, if you press 3 + 5 = and then keep pressing =, the calculator will keep counting by fives from the previous answer (the first sequence of numbers you wrote). This also works for the other three operations. A nice online calculator that can be projected in the classroom (and/or used with an interactive whiteboard) while children use their own handheld calculators can be found at www.online-calculator.com/full-screen-calculator.

2. Two Machines, One Job

Ron’s Recycle Shop started when Ron bought a used paper-shredding machine. Business was good, so Ron bought a new shredding machine. The old machine could shred a truckload
An Invitation to Do Mathematics

after products. For example, draw rectangles (or arrays) with a length and height of each of the factors (see Figure 2.3(a)), then draw the new rectangle (e.g., 8-by-6-unit rectangle). See how the rectangles compare.

You may prefer to think of multiplication as equal sets. For example, using stacks of chips, 7 × 7 is seven stacks with seven chips in each stack (set) (see Figure 2.3(b)). The expression 8 × 6 is represented by eight stacks of six (though six stacks of eight is a possible interpretation). See how the stacks for each expression compare.

3. One up, One Down

For Grades 1–3. When you add 7 + 7, you get 14. When you make the first number 1 more and the second number 1 less, you get the same answer:

\[
\uparrow \downarrow
\]

\[7 + 7 = 14\] and \[8 + 6 = 14\]

It works for 5 + 5 too:

\[
\uparrow \downarrow
\]

\[5 + 5 = 10\] and \[6 + 4 = 10\]

Does this work any time the numbers are the same? Does it work in other situations where the addends are not the same? Explore and develop your own conjectures.

For Grades 4–8. Does the one up, one down pattern apply to multiplication?

\[
\uparrow \downarrow
\]

\[7 \times 7 = 49\]

\[8 \times 6 = 48\]

\[
\uparrow \downarrow
\]

\[5 \times 5 = 25\]

\[6 \times 4 = 24\]

In these two multiplication examples, One Up, One Down resulted in an answer that is not equal, but is one less than the original problem. Does this work any time the original numbers are the same? Does it work in other products where the original numbers are not the same? Explore and develop your own conjectures.

STOP

Explore the multiplication problem, responding to the questions posed.

A Few Ideas. Multiplication is more complicated. Why? Use a physical model or picture to compare the before and after products. For example, draw rectangles (or arrays) with a length and height of each of the factors (see Figure 2.3(a)), then draw the new rectangle (e.g., 8-by-6-unit rectangle). See how the rectangles compare.

You may prefer to think of multiplication as equal sets. For example, using stacks of chips, 7 × 7 is seven stacks with seven chips in each stack (set) (see Figure 2.3(b)). The expression 8 × 6 is represented by eight stacks of six (though six stacks of eight is a possible interpretation). See how the stacks for each expression compare.

Figure 2.2 Cora’s solution to the paper-shredding problem.

Figure 2.3 Two physical ways to think about multiplication that might help in the exploration.
Chapter 2 Exploring What It Means to Know and Do Mathematics

STOP

Work with one or both of these approaches to gain insights and make conjectures.

Additional Patterns to Explore. Recall that doing mathematics includes the tendency to extend beyond the problem posed. This problem lends itself to many "what if?"s. Here are a few. If you have found other ones, great!

- Have you looked at how the first two numbers are related? For example, $7 \times 7$, $5 \times 5$, and $9 \times 9$ are all products with like factors. What if the product were two consecutive numbers (e.g., $8 \times 7$ or $13 \times 12$)? What if the factors differ by 2 or by 3?

- Think about adjusting by numbers other than one. What if you adjust two up and two down (e.g., $7 \times 7$ to $9 \times 5$)?

- What happens if you use big numbers instead of small ones (e.g., $30 \times 30$)?

- If both factors increase (i.e., one up, one up), is there a pattern?

Have you made some mathematical connections and conjectures in exploring this problem? In doing so you have hopefully felt a sense of accomplishment and excitement—one of the benefits of doing mathematics.

STOP

Think about the problem and what you know. Experiment.

A Few Ideas. Sometimes it is tough to get a feel for problems that are abstract or complex. In situations involving chance, find a way to experiment and see what happens. For this problem, you can make spinners using a drawing on paper, a paper clip, and a pencil. Put your pencil point through the loop of the clip and on the center of your spinner. Now you can spin the paper clip "pointer." Try at least 20 pairs of spins for each choice and keep track of what happens.

Consider these issues as you explore:

- For Susan’s choice (A then B), would it matter if she spun B first and then A? Why or why not?

- Explain why you think purple is more or less likely in one of the three cases compared to the other two. It sometimes helps to talk through what you have observed to come up with a way to apply some more precise reasoning.

STOP

Try these suggestions before reading on.

Strategy 1: Tree Diagrams. On spinner A, the four colors each have the same chance of coming up. You could make a tree diagram for A with four branches, and all the branches would have the same chance (see Figure 2.5). Spinner B has different-sized sections, leading to the following questions:

- What is the relationship between the blue region and each of the others?

- How could you make a tree diagram for B with each branch having the same chance?

- How can you add to the diagram for spinner A so that it represents spinning A twice in succession?

- Which branches on your diagram represent getting purple?

4. The Best Chance of Purple

Three students are spinning to "get purple" with two spinners, either by spinning first red and then blue or first blue and then red (see Figure 2.4). They may choose to spin each spinner once or one of the spinners twice. Mary chooses to spin twice on spinner A; John chooses to spin twice on spinner B; and Susan chooses to spin first on spinner A and then on spinner B. Who has the best chance of getting a red and a blue? [Lappan & Even, 1989, p. 17]

**Figure 2.4** You may spin A twice, B twice, or A then B. Which option gives you the best chance of spinning a red and a blue?

**Figure 2.5** A tree diagram for spinner A in Figure 2.4.
What Does It Mean to Learn Mathematics?

Now that you have had the chance to experience doing mathematics, you may have a series of questions: Can students solve such challenging tasks? Why take the time to solve these problems—isn’t it better to do a lot of shorter problems? Why should students be doing problems like this, especially if they are reluctant to do so? In other words, how does “doing mathematics” relate to student learning? The answer lies in learning theory and research on how people learn.

Learning theories have been developed through analysis of students (and adults) as they develop new understandings. Here we describe two theories (constructivism and sociocultural theory) that are most commonly used by researchers in mathematics education to understand how students learn mathematics. These theories are not competing, but are compatible (Norton & D’Ambrosio, 2008). Learning theories might be thought of as tools or lenses for interpreting how a person learns (Simon, 2009). For example, constructivism might be the best tool, or lens, for thinking about how a student might internalize an idea, and sociocultural theory might be a better tool for analyzing influence of the social/cultural aspects of the classroom.

Constructivism

Constructivism is rooted in Jean Piaget’s work, which was developed in the 1930s and translated to English in the 1950s. At the heart of constructivism is the notion that learners are not blank slates but rather creators (constructors) of their own learning. Integrated networks, or cognitive schemas, are both the product of constructing knowledge and the tools with which additional new knowledge can be constructed. As learning occurs, the networks are rearranged, added to, or otherwise modified. This is an active endeavor on the part of the learner (Baroody, 1987; Cobb, 1988; Fosnot, 1996; von Glasersfeld, 1990, 1996).

All people, all of the time, construct or give meaning to things they perceive or think about. As you read these words, you are giving meaning to them. Whether listening passively to a lecture or actively engaging in synthesizing findings in a project, your brain is applying prior knowledge (existing schemas) to make sense of the new information.

Strategy 2: Grids. Suppose that you had a square that represented all the possible outcomes for spinner A and a similar square for spinner B. Although there are many ways to divide a square into four equal parts, if you use lines going all in the same direction, you can make comparisons of all the outcomes of one event (one whole square) with the outcomes of another event (drawn on a different square). When the second event (in this case the second spin) follows the first event, make the lines on the second square go the opposite way from the lines on the first. Use transparencies and create squares to represent each spinner (see Figure 2.6). Place one over the other, and you will see 24 little sections.

Why are there six subdivisions for the spinner B square? What does each of the 24 little rectangles stand for? What sections would represent purple? Did 24 come into play in another strategy? Can you connect the tree diagram strategy to the rectangle strategy?

Where Are the Answers?

No answers or solutions are given in this text. How do you feel about that? What about the “right” answers? Are your answers correct? What makes the solution to any investigation “correct”?

In the classroom, the ready availability of the answer book or the teacher’s providing the solution or verifying that an answer is correct sends a clear message to students about doing mathematics: “Your job is to find the answers that the teacher already has.” In the real world of problem solving outside the classroom, there are no teachers with answers and no answer books. Doing mathematics includes using justification as a means of determining whether an answer is correct. The answer, then, to the question, is that the answers lie in your own reasoning and justification.

How could you make tree diagrams for John’s and Susan’s choices?

How do the tree diagrams relate to the spinners?

Tree diagrams are only one way to approach this. If the strategy makes sense to you, stop reading and solve the problem. If tree diagrams do not seem like a strategy you want to use, read on.
Through reflective thought (effort to connect existing ideas to new information), people modify their existing schemas to incorporate new ideas (Fosnot, 1996). This can happen in two ways—assimilation and accommodation. Assimilation occurs when a new concept “fits” with prior knowledge and the new information expands an existing network. Accommodation takes place when the new concept does not “fit” with the existing network (causing what Piaget called disequilibrium), so the brain revamps or replaces the existing schema. Though learning is constructed within the self, the classroom culture contributes to learning while the learner contributes to the culture in the classroom (Yackel & Cobb, 1996).

**Construction of Ideas.** To construct or build something in the physical world requires tools, materials, and effort. The tools we use to build understanding are our existing ideas and knowledge. The materials we use to build understanding may be things we see, hear, or touch, or our own thoughts and ideas. The effort required to connect new knowledge to old knowledge is reflective thought.

In Figure 2.7, blue and red dots are used as symbols for ideas. Consider the picture to be a small section of our cognitive makeup. The blue dots represent existing ideas. The lines joining the ideas represent our logical connections or relationships that have developed between and among ideas. The red dot is an emerging idea, one that is being constructed. Whatever existing ideas (blue dots) are used in the construction will be connected to the new idea (red dot) because those were the ideas that gave meaning to it. If a potentially relevant idea (blue dot) is not accessed by the learner when learning a new concept (red dot), then that potential connection will not be made.

**Sociocultural Theory**

In the same way that the work of Piaget relates to constructivism, the work of Lev Vygotsky, a Russian psychologist, has greatly influenced sociocultural theory. Vygotsky’s work also emerged in the 1920s and 1930s, but was not translated into English until the late 1970s. There are many concepts that these theories share (for example, the learning process as active meaning-seeking on the part of the learner), but sociocultural theory has several unique features. One is that mental processes exist between and among people in social learning settings, and that from these social settings the learner moves ideas into his or her own psychological realm (Forman, 2003).

Second, the way in which information is internalized (or learned) depends on whether it was within a learner’s zone of proximal development (ZPD) (Vygotsky, 1978). Simply put, the ZPD refers to a “range” of knowledge that may be out of reach for a person to learn on his or her own, but is accessible if the learner has support from peers or more knowledgeable others. “[T]he ZPD is not a physical space, but a symbolic space created through the interaction of learners with more knowledgeable others and the culture that precedes them” (Goos, 2004, p. 262). Researchers Cobb (1994) and Goos (2004) suggest that in a true mathematical community of learners there is something of a common ZPD that emerges across learners and there are also the ZPDs of individual learners.

Another major concept in sociocultural theory is semiotic mediation. Semiotic refers to the use of language, and other ways to convey cultural practices, such as diagrams, pictures, and actions visuals, and mediation means that these semiotics are exchanged between and among people. So, semiotic mediation is the “mechanism by which individual beliefs, attitudes, and goals are simultaneously affected and affect sociocultural practices and institutions” (Forman & McPhail, 1993, p. 134). In mathematics, semiotics include mathematical symbols (e.g., the equal sign), and it is through classroom interactions and activities that the meaning of these symbols are developed.

Social interaction is essential for mediation. The nature of the community of learners is affected by not just the culture the teacher creates, but the broader social and historical culture of the members of the classroom (Forman, 2003). In summary, from a sociocultural perspective, learning is dependent on the new knowledge falling within the ZPD of the learner (who must have access to the assistance),
and occurs through interactions that are influenced by tools of mediation (words, pictures, etc.) and the culture within and beyond the classroom.

**Implications for Teaching Mathematics**

It is not necessary to choose between a social constructivist theory that favors the views of Vygotsky and a cognitive constructivism built on the theories of Piaget (Cobb, 1994; Simon, 2009). In fact, when considering classroom practices that maximize opportunities to construct ideas, or to provide tools to promote mediation, they are quite similar. Classroom discussion based on students’ own ideas and solutions to problems is absolutely “foundational to children’s learning” (Wood & Turner-Vorbeck, 2001, p. 186).

Remember that learning theory is not a teaching strategy—theory informs teaching. This section outlines teaching strategies that are informed by constructivist and sociocultural perspectives. You will see these strategies revisited in Chapters 3 and 4, where a problem-based model for instruction is discussed, and in Section II, where you learn how to apply these ideas to specific areas of mathematics.

Importantly, if these strategies are grounded in how people learn, it means all people learn this way—students with special needs, English language learners, students who struggle, and students who are gifted. Too often, when teachers make adaptations and modifications for particular learners, they abandon these problem-based strategies for methods that involve fewer opportunities for students to connect ideas and build knowledge—thereby impeding, not supporting, learning.

**Build New Knowledge from Prior Knowledge.** Consider the following task.

Four children had 3 bags of M&Ms. They decided to open all 3 bags of candy and share the M&Ms fairly. There were 52 M&M candies in each bag. How many M&M candies did each child get? (Campbell & Johnson, 1995, pp. 35–36)

*Note:* You may want to select a nonfood context, such as decks of cards, or any culturally relevant or interesting item that would come in similar quantities.

**STOP**

Consider how you might introduce division to third graders and what your expectations might be for this problem as a teacher grounding your work in constructivist or sociocultural learning theory.

The student work samples in Figure 2.8 are from a classroom where students are asked to develop strategies for doing mathematics using their prior knowledge and explain their reasoning. From a constructivist and sociocultural perspective, this classroom culture allows students to access their prior knowledge, use cultural tools, and build new knowledge.

Marlena interpreted the first task as “How many sets of 4 can be made from 156?” She used facts that were either easy or available to her: 10 × 4 and 4 × 4. These totals she subtracted from 156 until she got to 100. This seemed to cue her to use 25 fours. She added her sets of 4 and knew the answer was 39 candies for each child. Marlena is using an equal subtraction approach using known multiplication facts. While this is not the most efficient approach, it demonstrates that Marlena understands the concept of division and, with the assistance of others, can move toward more efficient approaches.

Darrell’s approach reflects the sharing context of the problem. He formed four columns and distributed amounts mentally and orally as
he wrote the numbers. Darrell used a counting-up approach, first giving each student 20 M&Ms, seeing they could get more, distributed 5 to each, then 10, then singles until he reached the total. Like Marlena, Darrell used facts and procedures that he knew.

Note that this approach, in which students explore a problem and the mathematical ideas are later connected to that experience, is called a problem-based or inquiry approach. It is through inquiry that students are activating their own knowledge and trying to assimilate or accommodate (or internalize) new knowledge.

**Provide Opportunities to Talk about Mathematics.** Learning is enhanced when the learner is engaged with others working on the same ideas. A worthwhile goal is to create an environment in which students interact with each other and with you. The rich interaction in such a classroom allows students to engage in reflective thinking and to internalize concepts that may be out of reach without the interaction and input from peers and their teacher. In discussions with peers, students will be adapting and expanding on their existing networks of concepts.

**Build In Opportunities for Reflective Thought.** Classrooms need to provide structures and supports to help students make sense of mathematics in light of what they know. For a new idea you are teaching to be interconnected in a rich web of interrelated ideas, children must be mentally engaged. They must find the relevant ideas they possess and bring them to bear on the development of the new idea. In terms of the dots in Figure 2.7 we want to activate every blue dot students have that is related to the new red dot we want them to learn. Interestingly, this practice, grounded in learning theory, also has been established through research studies. Recall the research finding, stated earlier, that making mathematical relationships explicit is connected with improving student conceptual understanding (Hiebert & Grouws, 2007).

A key to getting students to be reflective is to engage them in interesting problems in which they use their prior knowledge as they search for solutions and create new ideas in the process. The problem-solving (inquiry) approach requires not just answers but also explanations and justifications for solutions.

**Encourage Multiple Approaches.** Teaching should provide opportunities for students to build connections between what they know and what they are learning. The student whose work is presented in Figure 2.9 may not understand the algorithm she is trying to use. If instead she were asked to use her own approach to find the difference, she might be able to get to a correct solution and build on her understanding of place value and subtraction.

Even learning a basic fact, like $7 \times 8$, can have better results if a teacher promotes multiple strategies. Imagine a class where students discuss and share clever ways to figure out the product. One student might think of 5 eights (40) and then 2 more eights (16) to equal 56. Another may have learned $7 \times 7$ (49) and added on 7 more to get 56. Still another might think “8 sevens” and take half of the sevens ($4 \times 7$) to get 28 and double 28 to get 56. A class discussion sharing these ideas brings to the fore a wide range of useful mathematical “dots” relating addition and multiplication concepts.

In contrast, facts such as $7 \times 8$ can be learned by rote (memorized). This knowledge is still constructed, but it is not connected to other knowledge. No blue dots! Rote learning can be thought of as a “weak construction” (Noddings, 1993). Students can recall it if they remember it, but if they forget, they don’t have $7 \times 8$ connected to other knowledge pieces that would allow them to redefine the fact.

**Engage Students in Productive Struggle.** Related to supporting multiple approaches, it is important to allow students the time to struggle with the mathematics they are exploring. As Piaget describes, learners are going to experience disequilibrium in developing new ideas. Let students know this disequilibrium is part of the process. Susan Carter, a National Board Certified Teacher who learned to engage her students in productive struggle, writes of her transformation,

I repeated the mantras of ineffective teachers: “This is too hard for them!” or “My kids just don’t have the background for this kind of assignment.” ... Imagine my heartbreak when I realized the disservice I was doing to my students, especially the ones who needed it most. By substituting a focus on happiness for a focus on engagement with the ideas, I deprived students of what they needed most: worthwhile mathematical tasks and the support to think through them. The more I challenged myself ... the closer I moved to an understanding of the necessity of struggle in learning.” (Carter, 2008, p. 135)

![Figure 2.9](image)
This is not just one teacher’s “aha”; this is one of the findings mentioned earlier as key to developing conceptual understanding (Hiebert & Grouws, 2007). This means redefining what we think of as “helping” students—rather than showing students how to do something, your role in helping students is to ask probing questions that keep students engaged in the productive struggle until they reach a solution. This communicates high expectations and maximizes students’ opportunities to learn with understanding.

**Treat Errors as Opportunities for Learning.** When students make errors, it can mean a misapplication of their prior knowledge in the new situation. Remember that from a constructivist perspective, the mind is sifting through what it knows in order to find useful approaches for the new situation. Knowing that children rarely give random responses (Ginsburg, 1977; Labinowicz, 1985) gives insight into addressing student misconceptions and helping students accommodate new learning. For example, students comparing decimals may incorrectly apply “rules” of whole numbers, such as “the more digits, the bigger the number” (Martinie, 2007; Resnick, Nesher, Leonard, Magone, Omanson, & Peled, 1989). Often one student’s misconception is shared by others in the class, and discussing the problem publicly can help other students understand (Hoffman, Breyfogle, & Dressler, 2009). This public negotiation of meaning allows students to construct deeper meaning for the mathematics.

Figure 2.9 is an example of a student incorrectly applying what she learned about regrouping. If the teacher tries to help the student by re-explaining the “right” way to do the problem, the student loses the opportunity to reflect on and correct her misconceptions. If the teacher instead asks the student to explain her regrouping process, the student must engage her reflective thought and think about what was regrouped and how to keep the number equivalent.

**Scaffold New Content.** The practice of scaffolding, often associated with sociocultural theory, is based on the idea that a task otherwise outside of a student’s ZPD can become accessible if it is carefully structured. For concepts completely new to students, the learning requires more structure or assistance, including the use of tools like manipulatives or more assistance from peers. As students become more comfortable with the content, the scaffolds are removed and the student becomes more independent. Scaffolding can provide support for those students who may not have a robust collection of “blue dots.”

**Honor Diversity.** Finally, and importantly, these theories emphasize that each learner is unique, with a different collection of prior knowledge and cultural experiences. Since new knowledge is built on existing knowledge and experience, effective teaching incorporates and builds on what the students bring to the classroom, honoring those experiences. Thus, lessons begin with eliciting prior experiences, and understandings and contexts for the lessons are selected based on students’ knowledge and experiences. Some students will not have all the “blue dots” they need—it is your job to provide experiences where those blue dots are developed and then connected to the concept being learned.

Classroom culture influences the individual learning of your students. As stated previously, you should support a range of approaches and strategies for doing mathematics. Students’ ideas should be valued and included in classroom discussion of the mathematics. This shift in practice, away from the teacher telling one way to do the problem, establishes a classroom culture where ideas are valued. This approach values the uniqueness of each individual.

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### What Does It Mean to Understand Mathematics?

Both constructivist and sociocultural theories emphasize the learner building connections (blue dots to the red dots) among existing and new ideas. So you might be asking, “What is it they should be learning and connecting?” Or “What are those red dots?” This section focuses on mathematics content required in today’s classrooms.

It is possible to say that we know something or we do not. That is, an idea is something that we either have or don’t have. Understanding is another matter. For example, most fifth graders know something about fractions. Given the fraction 1/2, they likely know how to read the fraction and can identify the 6 and 8 as the numerator and denominator, respectively. They know it is equivalent to 1/4 and that it is more than 1/2.

Students will have different understandings, however, of such concepts as what it means to be equivalent. They may know that 2/4 can be simplified to 1/2 but not understand that 1/2 and ½ represent identical quantities. Some may think that simplifying 2/4 to ½ makes it a smaller number. Some students will be able to create pictures or models to illustrate equivalent fractions or will have many examples of how ½ is used outside of class. In summary, there is a range of ideas that students often connect to their individualized understanding of a fraction—each student brings a different set of blue dots to his or her knowledge of what a fraction is.

Understanding can be defined as a measure of the quality and quantity of connections that an idea has with existing ideas. Understanding is not an all-or-nothing proposition. It depends on the existence of appropriate ideas and on the creation of new connections, varying with each person (Backhouse, Haggarty, Pirie, & Stratton, 1992; Davis, 1986; Hiebert & Carpenter, 1992).
Chapter 2 Exploring What It Means to Know and Do Mathematics

It is incorrect to say that a tool “illustrates” a concept. To illustrate implies showing. Technically, all that you actually see with your eyes is the physical object; only your mind can impose the mathematical relationship on the object (Suh, 2007b; Thompson, 1994).

Manipulatives can be a testing ground for emerging ideas. It is sometimes difficult for students (of all ages) to think about and test abstract relationships using only words or symbols. For example, students exploring the relationship between perimeter and area might use color tiles (squares of various colors), a geoboard (pegs on a grid) with rubber bands, or toothpicks to make the rectangles. A variety of tools should be accessible for students to select and use freely.

Examples of Tools. Physical materials or manipulatives in mathematics abound—from common objects such as lima

Relational Understanding

One way that we can think about understanding is that it exists along a continuum from a relational understanding—knowing what to do and why—to an instrumental understanding—doing something without understanding (see Figure 2.10). The two ends of this continuum were named by Richard Skemp (1978), an educational psychologist who has had a major influence on mathematics education.

In the \( \frac{2}{3} \) example, the student who can draw diagrams, give examples, find equivalencies, and approximate the size of \( \frac{2}{3} \) has an understanding toward the relational end of the continuum, while a student who only knows the names and a procedure for simplifying \( \frac{2}{3} \) to \( \frac{1}{2} \) has an understanding closer to the instrumental end of the continuum.

Multiple Representations. The more ways children are given to think about and test an emerging idea, the better chance they will correctly form and integrate it into a rich web of concepts and therefore develop a relational understanding. Figure 2.11 illustrates five representations for demonstrating an understanding of any topic (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). Lesh and colleagues have found that children who have difficulty translating a concept from one representation to another also have difficulty solving problems and understanding computations. Strengthening the ability to move between and among these representations improves student understanding and retention. Discussion of oral language, real-world situations, and written symbols is woven into this chapter, but here we elaborate on how manipulatives and models can help (or fail to help) children construct ideas.

Tools and Manipulatives. A tool for a mathematical concept refers to any object, picture, or drawing that represents the concept or onto which the relationship for that concept can be imposed. Manipulatives are physical objects that students and teachers can use to illustrate and discover mathematical concepts, whether made specifically for mathematics (e.g., connecting cubes) or for other purposes (e.g., buttons).

![Figure 2.10](image)

Understanding is a measure of the quality and quantity of connections that a new idea has with existing ideas. The greater the number of connections to a network of ideas, the better the understanding.

![Figure 2.11](image)

Five different representations of mathematical ideas. Translations between and within each can help develop new concepts.
beans and string to commercially produced materials such as wooden rods (e.g., Cuisenaire rods) and blocks (e.g., pattern blocks). Figure 2.12 shows six tools, each representing a different concept, giving only a glimpse into the many ways each manipulative can be used to support the development of mathematics concepts and procedures.

**STOP**

Consider each of the concepts and the corresponding model in Figure 2.12. Try to separate the physical tool from the relationship that you must impose on the tool in order to “see” the concept.

The examples in Figure 2.12 are models that can show the following concepts:

a. The concept of “6” is a relationship between sets that can be matched to the words one, two, three, four, five, or six. Changing a set of counters by adding one changes the relationship. The difference between the set of 6 and the set of 7 is the relationship “one more than.”

b. The concept of “measure of length” is a comparison. The length measure of an object is a comparison relationship of the length of the object to the length of the unit.

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**FIGURE 2.12** Examples of tools to illustrate mathematics concepts.

(a) Countable objects can be used to model “number” and related ideas such as “one more than.”

(b) "Length" involves a comparison of the length attribute of different objects. Rods can be used to measure length.

(c) "Rectangles" can be modeled on a dot grid. They involve length and spatial relationships.

(d) Base-ten concepts (ones, tens, hundreds) are frequently modeled with base 10 blocks. Sticks and bundles of sticks are also commonly used.

(e) "Chance" can be modeled by comparing outcomes of a spinner.

(f) "Positive" and "negative" integers can be modeled with arrows with different lengths and directions.
The concept of “rectangle” includes both spatial and length relationships. The opposite sides are of equal length and parallel, and the adjacent sides meet at right angles.

d. The concept of “hundred” is not in the larger square but in the relationship of that square to the strip (“ten”) and to the little square (“one”).

e. “Chance” is a relationship between the frequency of an event happening compared with all possible outcomes. The spinner can be used to create relative frequencies. These can be predicted by observing relationships of sections of the spinner.

f. The concept of a “negative integer” is based on the relationships of “magnitude” and “is the opposite of.” Negative quantities exist only in relation to positive quantities. Arrows on the number line model the opposite of relationship in terms of direction and size or magnitude relationship in terms of length.

Ineffective Use of Tools and Manipulatives. In addition to not making the connection between the model and the concept, there are other ways that models or manipulatives can be used ineffectively. One of the most widespread misuses occurs when the teacher tells students, “Do as I do.” There is a natural temptation to get out the materials and show children exactly how to use them. Children mimic the teacher’s directions, and it may even look as if they understand, but they could be just following what they see. It is just as possible to move blocks around mindlessly as it is to “invert and multiply” mindlessly. Neither promotes thinking about the development of concepts (Ball, 1992; Clements & Battista, 1990; Stein & Bovalino, 2001). For example, if you have carefully shown and explained how to get an answer to a multiplication problem with a set of base-ten blocks, then students may set up the blocks to get the answer but not focus on the patterns or processes that can be seen in modeling the problem with the blocks.

Conversely, leaving students with insufficient focus or guidance results in nonproductive and unsystematic investigation (Stein & Bovalino, 2001). Students may be engaged in conversations about the model they are using, but if they do not know what the mathematical goal is, the manipulative is not serving as a tool for developing the concept.

Technology-Based Tools. Technology provides another source of models and manipulatives. There are websites, such as the National Library of Virtual Manipulatives, that have a range of manipulatives available (e.g., geoboards, base-ten blocks, spinners, number lines). Virtual manipulatives are a good addition to physical models. In some cases, the electronic version allows users to interact with a manipulative in a way that is difficult or impossible to do with hands-on tools, and it may be accessed at home.

Students with physical disabilities may be better able to work with electronic versions of manipulatives.

It is important to include calculators as a tool. The calculator models a wide variety of numeric relationships by quickly and easily demonstrating the effects of these ideas. For example, you can skip-count by hundredths from 0.01 (press 0.01 \( \frac{1}{10} \), \( \frac{1}{100} \), \( \frac{1}{1000} \), . . . ) or from another beginning number such as 3 (press 3, 0.01 \( \frac{1}{10} \), \( \frac{1}{100} \), \( \frac{1}{1000} \), . . . ). How many presses of \( \frac{1}{10} \) are required to get from 3 to \( \frac{4}{2} \)?

Mathematics Proficiency

Much work has emerged since Skemp’s classic work emphasized the need for relational and instrumental understanding, based on the need to develop a robust understanding of mathematics. Mathematically proficient people exhibit certain behaviors and dispositions as they are “doing mathematics.” *Adding It Up* (National Research Council, 2001), an influential report on how students learn mathematics, describes five strands involved in being mathematically proficient: (1) conceptual understanding, (2) procedural fluency, (3) strategic competence, (4) adaptive reasoning, and (5) productive disposition. Figure 2.13 illustrates these interrelated and interwoven strands, providing a definition of each. These five proficiencies are the foundation for the Standards for Mathematical Practice described in the *Common Core State Standards* (CCSSO, 2010). The Standards for Mathematical Practices can be found in Table 1.2 on page 6.

Conceptual Understanding. Conceptual understanding is knowledge about the relationships or foundational ideas of a topic. Consider the task of adding 37 + 28. The conceptual understanding of this problem includes such ideas as this being a combining situation; that it could represent 37 people and then 28 more arriving; and that this is the same as 30 + 20 + 7 + 8, since you can take numbers apart, regroup, and still get the same sum. Additionally, students might understand that the value is larger than 50, but not much larger. (This relates to the Standards for Mathematical Practice in the *Common Core State Standards*: “1. Make sense of problems and persevere in solving them”; “7. Look for and make use of structure” [CCSSO, 2010].)

Procedural Fluency. Procedural fluency is knowledge and use of rules and procedures used in carrying out mathematical processes and also the symbolism used to represent mathematics. A student may choose to use the traditional algorithm (see Figure 2.14b) or employ an invented approach (see Figure 2.14c or (d)). A student who is procedurally fluent might move part of one number to another (see 2.14c) or use a counting-up strategy (see 2.14a). This choice will vary with the problem. He or she is flexible in ways to compute an answer. Note that the ability to
employ invented strategies, such as the ones described here, requires a conceptual understanding of place value and multiplication.

The ineffective practice of teaching procedures in the absence of conceptual understanding results in a lack of retention and increased errors. Think about the following problem: $40,005 - 39,996 = \_\_\_\_$. A student with weak procedural skills may launch into the standard algorithm, regrouping across zeros (this usually doesn’t go well), rather than notice that the number 39,996 is just 4 away from 40,000, and therefore notice that the difference between the two numbers is 9. Much research supports the fact that conceptual understanding is critical to developing procedural proficiency (Bransford et al., 2000; National Mathematics Advisory Panel, 2008; NCTM, 2000). The Principles and Standards Learning Principle states it well:

The alliance of factual knowledge, procedural proficiency, and conceptual understanding makes all three components usable in powerful ways. (p. 19)

Excerpt reprinted with permission from Principles and Standards for School Mathematics, copyright © 2000 by the National Council of Teachers of Mathematics.

Students can also have weak understanding of concepts—for example, only understanding the ideas when tied to a context. It is important to note that having deep conceptual and procedural understanding is important in having a relational understanding (Baroody, Feil, & Johnson, 2007). One way to explore all the interrelated ideas for a topic is to create a network or web of associations, as demonstrated in Figure 2.15 (page 28) for the concept of ratio. Note how much is involved in having a relational understanding.
Productive Disposition. What was your reaction when you read the problem about the two machines? Did you think, “I can’t remember the way to do this type of problem”? Or, did you think, “I can solve this, let me now think how”? The first response is the result of a history of learning math in which you were shown how to do things, rather than challenged to apply your own knowledge. The latter response is a productive disposition—a “can do” attitude. If you were committed to making sense of and solving those tasks, knowing that if you kept at it, you would get to a solution, then you have a productive disposition. This relates to the perseverance we just talked about in Chapter 1. What more important thing can we instill in students than a “can do” attitude? (This relates to the Standards for Mathematical Practice in the Common Core State Standards: “1. Make sense of problems and persevere in solving them”; “8. Look for and express regularity in repeated reasoning” [CCSSO, 2010].)

Strategic Competence. In solving Problems 1 through 4 earlier in the chapter, did you design a strategy? If it didn’t work, did you try something else? Perhaps you decided to draw a diagram or to fold paper to help you model the task. If you did any of these things, and if you changed out one strategy for a different one, then you were demonstrating strategic competence. Think of the value of this strand, not just in mathematics, but as a life skill. You have a problem; you need to figure out how you will solve it. If at first you don’t succeed, try, try again. (This relates to the Standards for Mathematical Practice in the Common Core State Standards: “4. Model with mathematics”; “5. Use appropriate tools strategically” [CCSSO, 2010].)

Adaptive Reasoning. When you finished one of the problems, did you wonder whether you had it right? Did you have a way of convincing yourself or your peer that it had to be correct? Conversely, did you head down a wrong path and realize it wasn’t working? This capacity to reflect on your work, evaluate it, and then adapt, as needed, is adaptive reasoning. (This relates to the Standards for Mathematical Practice in the Common Core State Standards: “2. Reason abstractly and quantitatively”; “3. Construct viable arguments and critique the reasoning of others”; “8. Look for and express regularity in repeated reasoning” [CCSSO, 2010].)

Benefits of Developing Mathematical Proficiency

To teach for mathematical proficiency requires a lot of effort. Concepts and connections develop over time, not in a day. Tasks must be strategically selected to help students build connections. The important benefits to be derived

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**FIGURE 2.15** Potential web of ideas that could contribute to the understanding of “ratio.”
from relational understanding make the effort not only worthwhile but also essential.

**Effective Learning of New Concepts and Procedures.** Recall what learning theory tells us—students are actively building on their existing knowledge. The more robust their understanding of a concept, the more connections students are building, and the more likely it is they can connect new ideas to the existing conceptual webs they have. Fraction knowledge and place-value knowledge come together to make decimal learning easier, and decimal concepts directly enhance an understanding of percentage concepts and operations. Without these and many other connections, children will need to learn each new piece of information they encounter as a separate, unrelated idea.

**Less to Remember.** When students learn in an instrumental manner, mathematics can seem like endless lists of isolated skills, concepts, rules, and symbols that must be refreshed regularly and often seem overwhelming to keep straight. Constructivists talk about teaching “big ideas” (Brooks & Brooks, 1993; Hiebert et al., 1996; Schifter & Fosnot, 1993). Big ideas are really just large networks of interrelated concepts. Frequently, the network is so well constructed that whole chunks of information are stored and retrieved as single entities rather than isolated bits. For example, knowledge of place value subsumes rules about lining up decimal points, ordering decimal numbers, moving decimal points to the right or left in decimal-percent conversions, rounding and estimating, and a host of other ideas.

**Increased Retention and Recall.** Memory is a process of retrieving information. Retrieval of information is more likely when you have the concept connected to an entire web of ideas. If what you need to recall doesn’t come to mind, reflecting on ideas that are related can usually lead you to the desired idea eventually. For example, if you forget the formula for surface area of a rectangular solid, reflecting on what it would look like if unfolded and spread out flat can help you remember that there are six rectangular faces in three pairs that are each the same size.

**Enhanced Problem-Solving Abilities.** The solution of novel problems requires transferring ideas learned in one context to new situations. When concepts are embedded in a rich network, transferability is significantly enhanced and, thus, so is problem solving (Schoenfeld, 1992). When students understand the relationship between a situation and a context, they are going to know when to use a particular approach to solve a problem. While many students may be able to do this with whole-number computation, once problems increase in difficulty and numbers move to rational numbers or unknowns, students without a relational understanding are not able to apply the skills they learned to solve new problems.

**Improved Attitudes and Beliefs.** Relational understanding has an affective as well as a cognitive effect. When ideas are well understood and make sense, the learner tends to develop a positive self-concept and a confidence in his or her ability to learn and understand mathematics. There is a definite feeling of “I can do this! I understand!” There is no reason to fear or to be in awe of knowledge learned relationally. At the other end of the continuum, instrumental understanding has the potential of producing mathematics anxiety, a real phenomenon that involves fear and avoidance behavior.

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It seems appropriate to close this chapter by connecting some dots, especially because the ideas represented here are the foundation for the approach to each topic in the content chapters. This chapter began with discussing what doing mathematics is and challenging you to do some mathematics. Each of these tasks offered opportunities to make connections between mathematics concepts—connecting the blue dots.

Second, you read about learning theory—the importance of having opportunities to connect the dots. The best learning opportunities, according to constructivism and sociocultural theories, are those that engage learners in using their own knowledge and experience to solve problems through social interactions and reflection. This is what you were asked to do in the four tasks. Did you learn something new about mathematics? Did you connect an idea that you had not previously connected?

Finally, you read about understanding—that having relational knowledge (knowledge in which blue dots are well connected) requires conceptual and procedural understanding as well as other proficiencies. The problems that you solved in the first section emphasized concepts and procedures while placing you in a position to use strategic competence, adaptive reasoning, and a productive disposition.

This chapter focused on connecting the dots between theory and practice—building a case that your teaching must focus on opportunities for students to develop their own networks of blue dots. As you plan and design instruction, you should constantly reflect on how to elicit prior knowledge by designing tasks that reflect the social and cultural backgrounds of students, to challenge students to think critically and creatively, and to include a comprehensive treatment of mathematics.
Chapter 2 Exploring What It Means to Know and Do Mathematics

**RESOURCES for Chapter 2**

**RECOMMENDED READINGS**

**Articles**

This article offers a great teaching strategy for nurturing relational thinking. Examples of the engaging "one, some, or none" activity are given for geometry, number, and algebra activities.

This is a wonderful teacher’s story of how she infused the constructivist notion of disequilibrium and the related idea of productive struggle to support learning in her first-grade class.

This article describes the many concepts related to division.

As the title implies, this is a great resource for connecting the NRC’s Mathematics Proficiencies (National Research Council, 2001) to teaching.

**Books**


Lampert reflects on her personal experiences in teaching fifth grade and shares with us her perspectives on the many issues and complexities of teaching. It is wonderfully written and easily accessed at any point in the book.

**ONLINE RESOURCES**

**Classic Problems**

A nice collection of well-known problems (“Train A leaves the station at . . .”) along with discussion, solutions, and extensions.

**Constructivism in the Classroom**

[http://mathforum.org/mathed/constructivism.html](http://mathforum.org/mathed/constructivism.html)  
Provided by the Math Forum, this page contains links to numerous sites concerning constructivism as well as articles written by researchers.

**Utah State University National Library of Virtual Manipulatives**

A robust collection of virtual manipulatives. Many do not have corresponding, hands-on counterparts. A great site to bookmark and use.

**DISCUSSION QUESTIONS**

1. What is mathematics and what does it mean to do mathematics? After reflecting on these questions, read the Standards for Mathematical Practice (Appendix A) from the Common Core State Standards and consider ways in which your answers to these questions connect or do not connect with the vision of doing mathematics communicated in these practices.

2. Examine the “Classroom Environment for Doing Mathematics” (p. 14). What questions, if any, does this list of features raise for you? Is there anything you think should be added to the list? Why? What are some explicit and implicit teacher moves that support the development of a classroom that engages in doing mathematics?

3. Consider all the different ways that third graders in a problem-based classroom might solve the following task:

   Four children had 3 bags of M&Ms. They decided to open all 3 bags of candy and share the M&Ms fairly. There were 52 M&M candies in each bag. How many M&M candies did each child get? (Campbell & Johnson, 1995, pp. 35–36)

4. Examine the student work on page 21. What do you notice about Marlena’s approach? Darrell’s approach? If you had the opportunity to ask either child questions to better understand her/his mathematical thinking, what would you ask? If you had the opportunity to ask either child questions to extend her/his mathematical thinking, what would you ask? Why?

**PROFESSIONAL DEVELOPMENT ACTIVITIES**

1. Pose the following task:

   Some people say that to add four consecutive numbers, you add the first and the last numbers and multiply by 2. Is this always true? How do you know? (Stoessiger & Edmunds, 1992)
As teachers work on this task, make the following tools available to them: colored tiles or linking cubes, graph paper, and a calculator. As teachers work, press them to engage in doing mathematics (see the verbs on page 14 or have teachers generate a list). Discuss the various approaches to proving the statement suggested. Encourage teachers to examine connections between the various solution strategies you select to share. Finally, ask teachers to connect both their actions/interactions while doing the mathematics and their discussion of the various ways of proving the statement to the Standards for Mathematical Practice.

2. Ask teachers to examine problems in curriculum materials and select a problem from an upcoming lesson that they think will be one that will naturally prompt multiple representations, as shown in Figure 2.11. As they plan the lesson, ask teachers to think about ways that they can encourage students to translate between the various representations. While the point is not to have every representation in every lesson, it is important to be mindful of the opportunities that are provided to students to connect models with written symbols, or real-world situations with pictures, so that they can flexibly move among these representations to solve problems.

Among the many tools on PDToolkit, the following tools provide support for developing ideas and teaching aligned with knowing and doing mathematics:

- Observation tools
  - Productive Classroom Culture
  - Conceptual and Procedural Knowledge
  - Mathematical Proficiency
- "One Up and One Down" Expanded Lesson
- "Assessing Mathematical Proficiency" (adaptation of Frayer Model) professional development Blackline Master
- IMAP Video #500, "Cathy’s Class," focusing on students sharing solutions to a problem-based task—an opportunity to observe the mathematical practices at play in a classroom

COACH/TEACHER LEADER CONSIDERATIONS

- The problems posed in the first section of this chapter are excellent for the workshop setting. “One Up, One Down” is a nonthreatening activity for teachers who may be intimidated by mathematics (consider starting with just the adding example, then posing the multiplication one). See PDToolkit for an Expanded Lesson on this activity. “Two Machines, One Job” is excellent for middle school as there is actually very little prior knowledge needed and proportional reasoning is emphasized.

- Making sense of mathematical proficiency is a significant task. It can be helpful to just pick one (from Adding It Up or from the Standards for Mathematical Practice) and discuss what a student who is proficient in that area can do. Then shift the conversation to what teacher moves support student development of that proficiency. Consider using the Mathematical Proficiency Observation (on PDToolkit) in a coaching cycle, using one or more of the items as a focus for conversation, data gathering, and reflection. Or meet with a teacher and decide on one to three Standards for Mathematical Practice she or he would like you to focus on, based on the teacher’s goal for the lesson.

- Strengthening students’ ability to move between and among the various representations of problems such as written symbols, manipulative models, oral language, pictures, and real-world situations is very important. Translation across representations improves student understanding and retention (see, for example, Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). Use the “Assessing Mathematical Proficiency” (adaptation of Frayer Model) professional development Blackline Master on PDToolkit to try this with students.