CHAPTER NINE

Priority Queues and Heapsort

MANY APPLICATIONS REQUIRE that we process records with keys in order, but not necessarily in full sorted order and not necessarily all at once. Often, we collect a set of records, then process the one with the largest key, then perhaps collect more records, then process the one with the current largest key, and so forth. An appropriate data structure in such an environment supports the operations of inserting a new element and deleting the largest element. Such a data structure is called a *priority queue*. Using priority queues is similar to using queues (remove the oldest) and stacks (remove the newest), but implementing them efficiently is more challenging. The priority queue is the most important example of the generalized queue ADT that we discussed in Section 4.7. In fact, the priority queue is a proper generalization of the stack and the queue, because we can implement these data structures with priority queues, using appropriate priority assignments (see Exercises 9.3 and 9.4).

Definition 9.1 *A* **priority queue** *is a data structure of items with keys which supports two basic operations: insert a new item, and remove the item with the largest key.*

Applications of priority queues include simulation systems, where the keys might correspond to event times, to be processed in chronological order; job scheduling in computer systems, where the keys might correspond to priorities indicating which users are to be served first; and numerical computations, where the keys might be computational errors, indicating that the largest should be dealt with first.

We can use any priority queue as the basis for a sorting algorithm by inserting all the records, then successively removing the largest to get the records in reverse order. Later on in this book, we shall see how to use priority queues as building blocks for more advanced algorithms. In Part 5, we shall see how priority queues are an appropriate abstraction for helping us understand the relationships among several fundamental graph-searching algorithms; and in Part 6, we shall develop a file-compression algorithm using routines from this chapter. These are but a few examples of the important role played by the priority queue as a basic tool in algorithm design.

In practice, priority queues are more complex than the simple definition just given, because there are several other operations that we may need to perform to maintain them under all the conditions that might arise when we are using them. Indeed, one of the main reasons that many priority-queue implementations are so useful is their flexibility in allowing client application programs to perform a variety of different operations on sets of records with keys. We want to build and maintain a data structure containing records with numerical keys (*priorities*) that supports some of the following operations:

- *Construct* a priority queue from N given items.
- *Insert* a new item.
- *Remove the maximum* item.
- *Change the priority* of an arbitrary specified item.
- *Remove* an arbitrary specified item.
- *Join* two priority queues into one large one.

If records can have duplicate keys, we take "maximum" to mean "any record with the largest key value." As with many data structures, we also need to add a standard *test if empty* operation and perhaps a *copy* (clone) operation to this set.

There is overlap among these operations, and it is sometimes convenient to define other, similar operations. For example, certain clients may need frequently to *find the maximum* item in the priority queue, without necessarily removing it. Or, we might have an operation to *replace the maximum* item with a new item. We could implement operations such as these using our two basic operations as building blocks: *Find the maximum* could be *remove the maximum* followed by *insert*, and *replace the maximum* could be either *insert* followed by *remove the maximum* or *remove the maximum* followed by *insert*. We

Program 9.1 Basic priority-queue ADT

This interface defines operations for the simplest type of priority queue: initialize, test if empty, add a new item, remove the largest item. Elementary implementations of these methods using arrays and linked lists can require linear time in the worst case, but we shall see implementations in this chapter where all operations are guaranteed to run in time at most proportional to the logarithm of the number of items in the queue. The constructor's parameter specifies the maximum number of items expected in the queue and may be ignored by some implementations.

```
class PQ // ADT interface
  { // implementations and private members hidden
    PQ(int)
    boolean empty()
    void insert(ITEM)
    ITEM getmax()
  };
```

normally get more efficient code, however, by implementing such operations directly, provided that they are needed and precisely specified. Precise specification is not always as straightforward as it might seem. For example, the two options just given for *replace the maximum* are quite different: the former always makes the priority queue grow temporarily by one item, and the latter always puts the new item on the queue. Similarly, the *change priority* operation could be implemented as a *remove* followed by an *insert*, and *construct* could be implemented with repeated uses of *insert*.

For some applications, it might be slightly more convenient to switch around to work with the *minimum*, rather than with the maximum. We stick primarily with priority queues that are oriented toward accessing the maximum key. When we do need the other kind, we shall refer to it (a priority queue that allows us to *remove the minimum* item) as a *minimum-oriented* priority queue.

The priority queue is a prototypical *abstract data type (ADT)* (see Chapter 4): It represents a well-defined set of operations on data, and it provides a convenient abstraction that allows us to separate applications programs (clients) from various implementations that we will consider in this chapter. The interface given in Program 9.1 defines the most basic priority-queue operations; we shall consider a more complete interface in Section 9.5. Strictly speaking, different subsets of the various operations that we might want to include lead to different abstract data structures, but the priority queue is essentially characterized by the *remove-the-maximum* and *insert* operations, so we shall focus on them.

Different implementations of priority queues afford different performance characteristics for the various operations to be performed, and different applications need efficient performance for different sets of operations. Indeed, performance differences are, in principle, the *only* differences that can arise in the abstract-data-type concept. This situation leads to cost tradeoffs. In this chapter, we consider a variety of ways of approaching these cost tradeoffs, nearly reaching the ideal of being able to perform the *remove the maximum* operation in logarithmic time and all the other operations in constant time.

First, in Section 9.1, we illustrate this point by discussing a few elementary data structures for implementing priority queues. Next, in Sections 9.2 through 9.4, we concentrate on a classical data structure called the *heap*, which allows efficient implementations of all the operations but *join*. In Section 9.4, we also look at an important sorting algorithm that follows naturally from these implementations. In Sections 9.5 and 9.6, we look in more detail at some of the problems involved in developing complete priority-queue ADTs. Finally, in Section 9.7, we examine a more advanced data structure, called the *binomial queue*, that we use to implement all the operations (including *join*) in worst-case logarithmic time.

During our study of all these various data structures, we shall bear in mind both the basic tradeoffs dictated by linked versus sequential memory allocation (as introduced in Chapter 3) and the problems involved with making packages usable by applications programs. In particular, some of the advanced algorithms that appear later in this book are client programs that make use of priority queues.

Exercises

▷ **9.1** A letter means *insert* and an asterisk means *remove the maximum* in the sequence

P R I O * R * * I * T * Y * * * Q U E * * * U * E.

Give the sequence of values returned by the *remove the maximum* operations.

▷ **9.2** Add to the conventions of Exercise 9.1 a plus sign to mean *join* and parentheses to delimit the priority queue created by the operations within them. Give the contents of the priority queue after the sequence

(((P R I O *) + (R * I T * Y *)) * * *) + (Q U E * * * U * E).

∘ **9.3** Explain how to use a priority queue ADT to implement a stack ADT.

∘ **9.4** Explain how to use a priority queue ADT to implement a queue ADT.

9.1 Elementary Implementations

The basic data structures that we discussed in Chapter 3 provide us with numerous options for implementing priority queues. Program 9.2 is an implementation that uses an unordered array as the underlying data structure. The *find the maximum* operation is implemented by scanning the array to find the maximum, then exchanging the maximum item with the last item and decrementing the queue size. Figure 9.1 shows the contents of the array for a sample sequence of operations. This basic implementation corresponds to similar implementations that we saw in Chapter 4 for stacks and queues (see Programs 4.7 and 4.17) and is useful for small queues. The significant difference has to do with performance. For stacks and queues, we were able to develop implementations of all the operations that take constant time; for priority queues, it is easy to find implementations where *either* the *insert* or the *remove the maximum* operations takes constant time, but finding an implementation where *both* operations will be fast is a more difficult task, and it is the subject of this chapter.

We can use unordered or ordered sequences, implemented as linked lists or as arrays. The basic tradeoff between leaving the items unordered and keeping them in order is that maintaining an ordered sequence allows for constant-time *remove the maximum* and *find the maximum* but might mean going through the whole list for *insert*, whereas an unordered sequence allows a constant-time *insert* but might mean going through the whole sequence for *remove the maximum* and *find the maximum*. The unordered sequence is the prototypical *lazy* approach to this problem, where we defer doing work until necessary (to find the maximum); the ordered sequence is the prototypical *eager* approach to the problem, where we do as much work as we can up front (keep the list sorted on insertion) to make later operations efficient. We can use an array or linked-list representation in either case, with the basic tradeoff that the (doubly) linked list allows a

B		B
E		B E
*	E	B
S		B S
T		B S T
I		B S T I
*	T	B S I
N		L S I N
*	S	B N I
F		B N I F
I		B N I F I
R		B N I F I R
*	R	B N I F I
S		B N I F I S
T		B N I F I S T
*	T	B N I F I S
*	S	B N I F I
O		B N I F I O
U		B N I F I O U
*	U	B N I F I O
T		B N I F I O T
*	T	B N I F I O
*	O	B N I F I
*	N	B I I F
*	I	B F I
*	I	B F
*	F	B
*	B	

Figure 9.1
Priority-queue example (unordered array representation)

This sequence shows the result of the sequence of operations in the left column (top to bottom), where a letter denotes insert *and an asterisk denotes* remove the maximum. *Each line displays the operation, the letter removed for the remove-the-maximum operations, and the contents of the array after the operation.*

Program 9.2 Array implementation of a priority queue

This implementation, which may be compared with the array implementations for stacks and queues that we considered in Chapter 4 (see Programs 4.7 and 4.17), keeps the items in an unordered array. Items are added to and removed from the end of the array, as in a stack.

```
class PQ
  {
    static boolean less(ITEM v, ITEM w)
      { return v.less(w); }
    static void exch(ITEM[] a, int i, int j)
      { ITEM t = a[i]; a[i] = a[j]; a[j] = t; }
    private ITEM[] pq;
    private int N;
    PQ(int maxN)
      { pq = new ITEM[maxN]; N = 0; }
    boolean empty()
      { return N == 0; }
    void insert(ITEM item)
      { pq[N++] = item; }
    ITEM getmax()
      { int max = 0;
        for (int j = 1; j < N; j++)
          if (less(pq[max], pq[j])) max = j;
        exch(pq, max, N-1);
        return pq[--N];
      }
  };
```

constant-time *remove* (and, in the unordered case, *join*), but requires more space for the links.

The worst-case costs of the various operations (within a constant factor) on a priority queue of size N for various implementations are summarized in Table 9.1.

Developing a full implementation requires paying careful attention to the interface—particularly to how client programs access nodes for the *remove* and *change priority* operations, and how they access priority queues themselves as data types for the *join* operation. These

Table 9.1 Worst-case costs of priority-queue operations

Implementations of the priority queue ADT have widely varying performance characteristics, as indicated in this table of the worst-case time (within a constant factor for large N) for various methods. Elementary methods (first four lines) require constant time for some operations and linear time for others; more advanced methods guarantee logarithmic- or constant-time performance for most or all operations.

	insert	*remove maximum*	*remove*	*find maximum*	*change priority*	*join*
ordered array	N	1	N	1	N	N
ordered list	N	1	1	1	N	N
unordered array	1	N	1	N	1	N
unordered list	1	N	1	N	1	1
heap	$\lg N$	$\lg N$	$\lg N$	1	$\lg N$	N
binomial queue	$\lg N$	$\lg N$	$\lg N$	$\lg N$	$\lg N$	$\lg N$
best in theory	1	$\lg N$	$\lg N$	1	1	1

issues are discussed in Sections 9.4 and 9.7, where two full implementations are given: one using doubly linked unordered lists, and another using binomial queues.

The running time of a client program using priority queues depends not just on the keys but also on the mix of the various operations. It is wise to keep in mind the simple implementations because they often can outperform more complicated methods in many practical situations. For example, the unordered-list implementation might be appropriate in an application where only a few *remove the maximum* operations are performed, as opposed to a huge number of insertions, whereas an ordered list would be appropriate if a huge number of *find the maximum* operations are involved, or if the items inserted tend to be larger than those already in the priority queue.

Exercises

▷ **9.5** Criticize the following idea: To implement *find the maximum* in constant time, why not keep track of the maximum value inserted so far, then return that value for *find the maximum*?

▷ **9.6** Give the contents of the array after the execution of the sequence of operations depicted in Figure 9.1.

9.7 Provide an implementation for the basic priority-queue interface that uses an ordered array for the underlying data structure.

9.8 Provide an implementation for the basic priority-queue interface that uses an unordered linked list for the underlying data structure. *Hint*: See Programs 4.8 and 4.16.

9.9 Provide an implementation for the basic priority-queue interface that uses an ordered linked list for the underlying data structure. *Hint*: See Program 3.11.

∘ **9.10** Consider a lazy implementation where the list is ordered only when a *remove the maximum* or a *find the maximum* operation is performed. Insertions since the previous sort are kept on a separate list, then are sorted and merged in when necessary. Discuss advantages of such an implementation over the elementary implementations based on unordered and ordered lists.

• **9.11** Write a performance driver client program that uses `insert` to fill a priority queue, then uses `getmax` to remove half the keys, then uses `insert` to fill it up again, then uses `getmax` to remove all the keys, doing so multiple times on random sequences of keys of various lengths ranging from small to large; measures the time taken for each run; and prints out or plots the average running times.

• **9.12** Write a performance driver client program that uses `insert` to fill a priority queue, then does as many `getmax` and `insert` operations as it can do in 1 second, doing so multiple times on random sequences of keys of various lengths ranging from small to large; and prints out or plots the average number of `getmax` operations it was able to do.

9.13 Use your client program from Exercise 9.12 to compare the unordered-array implementation in Program 9.2 with your unordered-list implementation from Exercise 9.8.

9.14 Use your client program from Exercise 9.12 to compare your ordered-array and ordered-list implementations from Exercises 9.7 and 9.9.

• **9.15** Write an exercise driver client program that uses the methods in our priority-queue interface Program 9.1 on difficult or pathological cases that might turn up in practical applications. Simple examples include keys that are already in order, keys in reverse order, all keys the same, and sequences of keys having only two distinct values.

9.16 (This exercise is 24 exercises in disguise.) Justify the worst-case bounds for the four elementary implementations that are given in Table 9.1, by reference to the implementation in Program 9.2 and your implementations from Exercises 9.7 through 9.9 for *insert* and *remove the maximum*; and by informally describing the methods for the other operations. For *remove*, *change priority*, and *join*, assume that you have a handle that gives you direct access to the referent.

9.2 Heap Data Structure

The main topic of this chapter is a simple data structure called the *heap* that can efficiently support the basic priority-queue operations. In a heap, the records are stored in an array such that each key is guaranteed to be larger than the keys at two other specific positions. In turn, each of those keys must be larger than two more keys, and so forth. This ordering is easy to see if we view the keys as being in a binary tree structure with edges from each key to the two keys known to be smaller.

Definition 9.2 *A tree is* **heap-ordered** *if the key in each node is larger than or equal to the keys in all of that node's children (if any). Equivalently, the key in each node of a heap-ordered tree is smaller than or equal to the key in that node's parent (if any).*

Property 9.1 *No node in a heap-ordered tree has a key larger than the key at the root.*

We could impose the heap-ordering restriction on any tree. It is particularly convenient, however, to use a *complete binary tree*. Recall from Chapter 3 that we can draw such a structure by placing the root node and then proceeding down the page and from left to right, connecting two nodes beneath each node on the previous level until N nodes have been placed. We can represent complete binary trees sequentially within an array by simply putting the root at position 1, its children at positions 2 and 3, the nodes at the next level in positions 4, 5, 6 and 7, and so on, as illustrated in Figure 9.2.

Definition 9.3 *A* **heap** *is a set of nodes with keys arranged in a complete heap-ordered binary tree, represented as an array.*

We could use a linked representation for heap-ordered trees, but complete trees provide us with the opportunity to use a compact array representation where we can easily get from a node to its parent and children without needing to maintain explicit links. The parent of the node in position i is in position $\lfloor i/2 \rfloor$, and, conversely, the two children of the node in position i are in positions $2i$ and $2i + 1$. This arrangement makes traversal of such a tree even easier than if the tree were implemented with a linked representation, because, in a linked representation, we would need to have three links associated with each key to allow travel up and down the tree (each element would have

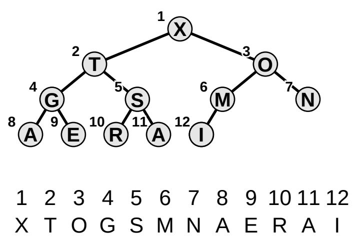


Figure 9.2
Array representation of a heap-ordered complete binary tree

Considering the element in position $\lfloor i/2 \rfloor$ in an array to be the parent of the element in position i, for $2 \le i \le N$ (or, equivalently, considering the ith element to be the parent of the $2i$th element and the $(2i+1)$st element), corresponds to a convenient representation of the elements as a tree. This correspondence is equivalent to numbering the nodes in a complete binary tree (with nodes on the bottom as far left as possible) in level order. A tree is heap-ordered if the key in any given node is greater than or equal to the keys of that node's children. A heap is an array representation of a complete heap-ordered binary tree. The ith element in a heap is larger than or equal to both the $2i$th and the $(2i+1)$st elements.

one pointer to its parent and one to each child). Complete binary trees represented as arrays are rigid structures, but they have just enough flexibility to allow us to implement efficient priority-queue algorithms.

We shall see in Section 9.3 that we can use heaps to implement all the priority-queue operations (except *join*) such that they require logarithmic time in the worst case. The implementations all operate along some path inside the heap (moving from parent to child toward the bottom or from child to parent toward the top, but not switching directions). As we discussed in Chapter 3, all paths in a complete tree of N nodes have about $\lg N$ nodes on them: there are about $N/2$ nodes on the bottom, $N/4$ nodes with children on the bottom, $N/8$ nodes with grandchildren on the bottom, and so forth. Each generation has about one-half as many nodes as the next, and there are at most $\lg N$ generations.

We can also use explicit linked representations of tree structures to develop efficient implementations of the priority-queue operations. Examples include triply linked heap-ordered complete trees (see Section 9.5), tournaments (see Program 5.19), and binomial queues (see Section 9.7). As with simple stacks and queues, one important reason to consider linked representations is that they free us from having to know the maximum queue size ahead of time, as is required with an array representation. In certain situations, we also can make use of the flexibility provided by linked structures to develop efficient algorithms. Readers who are inexperienced with using explicit tree structures are encouraged to read Chapter 12 to learn basic methods for the even more important symbol-table ADT implementation before tackling the linked tree representations discussed in the exercises in this chapter and in Section 9.7. However, careful consideration of linked structures can be reserved for a second reading, because our primary topic in this chapter is the heap (linkless array representation of the heap-ordered complete tree).

Exercises

▷ **9.17** Is an array that is sorted in descending order a heap?

○ **9.18** The largest element in a heap must appear in position 1, and the second largest element must be in position 2 or position 3. Give the list of positions in a heap of size 15 where the kth largest element (*i*) can appear, and (*ii*) cannot appear, for $k = 2, 3, 4$ (assuming the values to be distinct).

• **9.19** Answer Exercise 9.18 for general k, as a function of N, the heap size.

• 9.20 Answer Exercises 9.18 and 9.19 for the *k*th *smallest* element.

9.3 Algorithms on Heaps

The priority-queue algorithms on heaps all work by first making a simple modification that could violate the heap condition, then traveling through the heap, modifying the heap as required to ensure that the heap condition is satisfied everywhere. This process is sometimes called *heapifying,* or just *fixing* the heap. There are two cases. When the priority of some node is increased (or a new node is added at the bottom of a heap), we have to travel *up* the heap to restore the heap condition. When the priority of some node is decreased (for example, if we replace the node at the root with a new node), we have to travel *down* the heap to restore the heap condition. First, we consider how to implement these two basic methods; then, we see how to use them to implement the various priority-queue operations.

If the heap property is violated because a node's key becomes larger than that node's parent's key, then we can make progress toward fixing the violation by exchanging the node with its parent. After the exchange, the node is larger than both its children (one is the old parent, and the other is smaller than the old parent because it was a child of that node) but may still be larger than its parent. We can fix that violation in the same way, and so forth, moving up the heap until we reach a node with a larger key, or the root. An example of this process is shown in Figure 9.3. The code is straightforward, based on the notion that the parent of the node at position `k` in a heap is at position `k/2`. Program 9.3 is an implementation of a method that restores a possible violation due to increased priority at a given node in a heap by moving up the heap.

If the heap property is violated because a node's key becomes smaller than one or both of that node's childrens' keys, then we can make progress toward fixing the violation by exchanging the node with the larger of its two children. This switch may cause a violation at the child; we fix that violation in the same way, and so forth, moving down the heap until we reach a node with both children smaller, or the bottom. An example of this process is shown in Figure 9.4. The code again follows directly from the fact that the children of the node at position `k` in a heap are at positions `2k` and `2k+1`. Program 9.4 is an

Program 9.3 Bottom-up heapify

To restore the heap condition when an item's priority is increased, we move up the heap, exchanging the item at position `k` with its parent (at position `k/2`) if necessary, continuing as long as the item at position `k/2` is less than the node at position `k` or until we reach the top of the heap. The methods `less` and `exch` compare and exchange (respectively) the items at the heap indices specified by their parameters (see Program 9.5 for implementations).

```
private void swim(int k)
  {
    while (k > 1 && less(k/2, k))
      { exch(k, k/2); k = k/2; }
  }
```

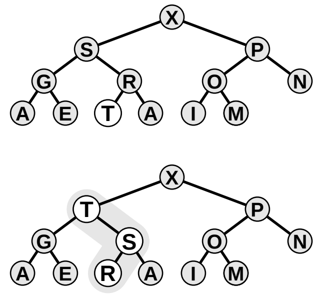


Figure 9.3
Bottom-up heapify (swim)

The tree depicted on the top is heap-ordered except for the node T *on the bottom level. If we exchange* T *with its parent, the tree is heap-ordered, except possibly that* T *may be larger than its new parent. Continuing to exchange* T *with its parent until we encounter the root or a node on the path from* T *to the root that is larger than* T*, we can establish the heap condition for the whole tree. We can use this procedure as the basis for the* insert *operation on heaps in order to reestablish the heap condition after adding a new element to a heap (at the rightmost position on the bottom level, starting a new level if necessary).*

implementation of a method that restores a possible violation due to increased priority at a given node in a heap by moving down the heap. This method needs to know the size of the heap (`N`) in order to be able to test when it has reached the bottom.

These two operations are independent of the way that the tree structure is represented, as long as we can access the parent (for the bottom-up method) and the children (for the top-down method) of any node. For the bottom-up method, we move up the tree, exchanging the key in the given node with the key in its parent until we reach the root or a parent with a larger (or equal) key. For the top-down method, we move down the tree, exchanging the key in the given node with the largest key among that node's children, moving down to that child, and continuing down the tree until we reach the bottom or a point where no child has a larger key. Generalized in this way, these operations apply not just to complete binary trees but also to any tree structure. Advanced priority-queue algorithms usually use more general tree structures but rely on these same basic operations to maintain access to the largest key in the structure, at the top.

If we imagine the heap to represent a corporate hierarchy, with each of the children of a node representing subordinates (and the parent representing the immediate superior), then these operations have amusing interpretations. The bottom-up method corresponds to a promising new manager arriving on the scene, being promoted up

Program 9.4 Top-down heapify

To restore the heap condition when a node's priority is decreased, we move down the heap, exchanging the node at position `k` with the larger of that node's two children if necessary and stopping when the node at `k` is not smaller than either child or the bottom is reached. Note that if `N` is even and `k` is `N/2`, then the node at `k` has only one child—this case must be treated properly!

The inner loop in this program has two distinct exits: one for the case that the bottom of the heap is hit, and another for the case that the heap condition is satisfied somewhere in the interior of the heap. It is a prototypical example of the need for the `break` construct.

```
private void sink(int k, int N)
  {
    while (2*k <= N)
      { int j = 2*k;
        if (j < N && less(j, j+1)) j++;
        if (!less(k, j)) break;
        exch(k, j); k = j;
      }
  }
```

the chain of command (by exchanging jobs with any lower-qualified boss) until the new person encounters a higher-qualified boss. Mixing methaphors, we also think about the new arrival having to *swim* to the surface. The top-down method is analogous to the situation when the president of the company is replaced by someone less qualified. If the president's most powerful subordinate is stronger than the new person, they exchange jobs, and we move down the chain of command, demoting the new person and promoting others until the level of competence of the new person is reached, where there is no higher-qualified subordinate (this idealized scenario is rarely seen in the real world). Again mixing metaphors, we also think about the new person having to *sink* to the bottom.

These two basic operations allow efficient implementation of the basic priority-queue ADT, as given in Program 9.5. With the priority queue represented as a heap-ordered array, using the *insert* operation amounts to adding the new element at the end and moving that element up through the heap to restore the heap condition; the *remove the*

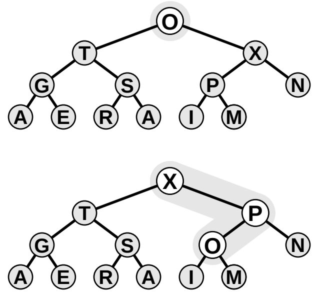


Figure 9.4
Top-down heapify (sink)

The tree depicted on the top is heap-ordered, except at the root. If we exchange the O *with the larger of its two children (*X*), the tree is heap-ordered, except at the subtree rooted at* O*. Continuing to exchange* O *with the larger of its two children until we reach the bottom of the heap or a point where* O *is larger than both its children, we can establish the heap condition for the whole tree. We can use this procedure as the basis for the* remove the maximum *operation on heaps in order to reestablish the heap condition after replacing the key at the root with the rightmost key on the bottom level.*

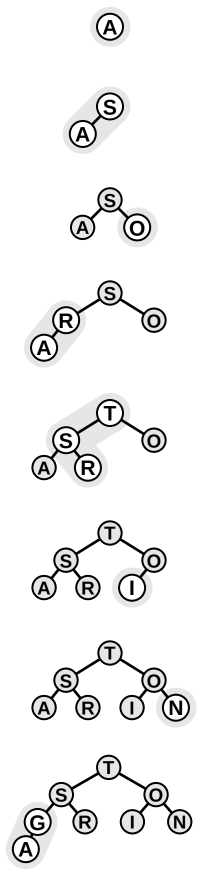


Figure 9.5
Top-down heap construction

This sequence depicts the insertion of the keys A S O R T I N G *into an initially empty heap. New items are added to the heap at the bottom, moving from left to right on the bottom level. Each insertion affects only the nodes on the path between the insertion point and the root, so the cost is proportional to the logarithm of the size of the heap in the worst case.*

Program 9.5 Heap-based priority queue

To implement `insert`, we increment `N`, add the new element at the end, then use `swim` to restore the heap condition. For `getmax` we take the value to be returned from `pq[1]`, then decrement the size of the heap by moving `pq[N]` to `pq[1]` and using `sink` to restore the heap condition. The first position in the array, `pq[0]`, is not used.

```
class PQ
  {
    private boolean less(int i, int j)
      { return pq[i].less(pq[j]); }
    private void exch(int i, int j)
      { ITEM t = pq[i]; pq[i] = pq[j]; pq[j] = t; }
    private void swim(int k)
      // Program 9.3
    private void sink(int k, int N)
      // Program 9.4
    private ITEM[] pq;
    private int N;
    PQ(int maxN)
      { pq = new ITEM[maxN+1]; N = 0; }
    boolean empty()
      { return N == 0; }
    void insert(ITEM v)
      { pq[++N] = v; swim(N); }
    ITEM getmax()
      { exch(1, N); sink(1, N-1); return pq[N--]; }
  };
```

maximum operation amounts to taking the largest value off the top, then putting in the item from the end of the heap at the top and moving it down through the array to restore the heap condition.

Property 9.2 *The* **insert** *and* **remove the maximum** *operations for the priority queue abstract data type can be implemented with heap-ordered trees such that* **insert** *requires no more than* $\lg N$ *comparisons and* **remove the maximum** *no more than* $2 \lg N$ *comparisons, when performed on an* N*-item queue.*

Both operations involve moving along a path between the root and the bottom of the heap, and no path in a heap of size N includes more than $\lg N$ elements (see, for example, Property 5.8 and Exercise 5.77). The *remove the maximum* operation requires two comparisons for each node: one to find the child with the larger key, the other to decide whether that child needs to be promoted. ■

Figures 9.5 and 9.6 show an example in which we construct a heap by inserting items one by one into an initially empty heap. In the array representation that we have been using for the heap, this process corresponds to heap ordering the array by moving sequentially through the array, considering the size of the heap to grow by 1 each time that we move to a new item, and using `swim` to restore the heap order. The process takes time proportional to $N \log N$ in the worst case (if each new item is the largest seen so far, it travels all the way up the heap), but it turns out to take only linear time on the average (a random new item tends to travel up only a few levels). In Section 9.4 we shall see a way to construct a heap (to heap order an array) in linear worst-case time.

The basic `swim` and `sink` operations in Programs 9.3 and 9.4 also allow direct implementation for the *change priority* and *remove* operations. To change the priority of an item somewhere in the middle of the heap, we use `swim` to move up the heap if the priority is increased, and `sink` to go down the heap if the priority is decreased. Full implementations of such operations, which refer to specific data items, make sense only if a handle is maintained for each item to that item's place in the data structure. In order to do so, we need to define an ADT for that purpose. We shall consider such an ADT and corresponding implementations in detail in Sections 9.5 through 9.7.

Property 9.3 *The* **change priority**, **remove**, *and* **replace the maximum** *operations for the priority queue abstract data type can be implemented with heap-ordered trees such that no more than $2 \lg N$ comparisons are required for any operation on an N-item queue.*

Since they require handles to items, we defer considering implementations that support these operations to Section 9.6 (see Program 9.12 and Figure 9.14). They all involve moving along one path in the heap, perhaps all the way from top to bottom or from bottom to top in the worst case. ■

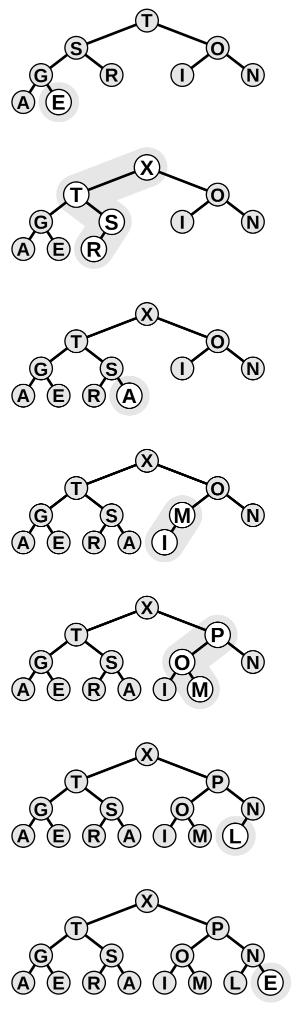


Figure 9.6
Top-down heap construction (continued)

This sequence depicts insertion of the keys E X A M P L E *into the heap started in Figure 9.5. The total cost of constructing a heap of size N is less than*

$$\lg 1 + \lg 2 + \ldots + \lg N,$$

which is less than $N \lg N$.

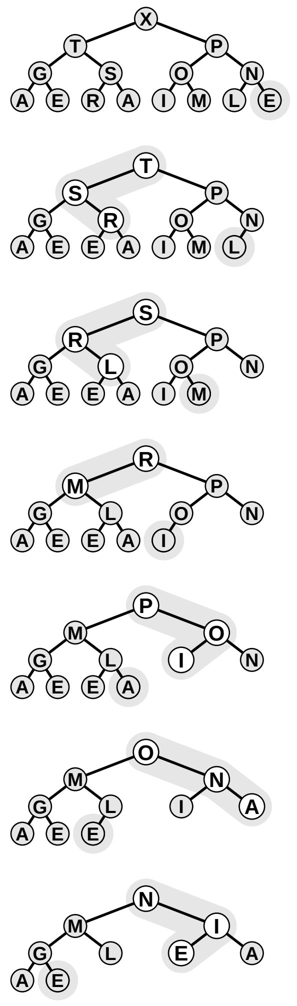


Figure 9.7
Sorting from a heap

After replacing the largest element in the heap by the rightmost element on the bottom level, we can restore the heap order by sifting down along a path from the root to the bottom.

Program 9.6 Sorting with a priority queue

This class uses our priority-queue ADT to implement the standard `Sort` class that was introduced in Program 6.3.

To sort a subarray `a[l]`, ..., `a[r]`, we construct a priority queue with enough capacity to hold all of its items, use `insert` to put all the items on the priority queue, and then use `getmax` to remove them, in decreasing order. This sorting algorithm runs in time proportional to $N \lg N$ but uses extra space proportional to the number of items to be sorted (for the priority queue).

```
class Sort
  {
    static void sort(ITEM[] a, int l, int r)
      { PQsort(a, l, r); }
    static void PQsort(ITEM[] a, int l, int r)
      { int k;
        PQ pq = new PQ(r-l+1);
        for (k = l; k <= r; k++)
          pq.insert(a[k]);
        for (k = r; k >= l; k--)
          a[k] = pq.getmax();
      }
  }
```

Note carefully that the *join* operation is not included on this list. Combining two priority queues efficiently seems to require a much more sophisticated data structure. We shall consider such a data structure in detail in Section 9.7. Otherwise, the simple heap-based method given here suffices for a broad variety of applications. It uses minimal extra space and is guaranteed to run efficiently *except* in the presence of frequent and large *join* operations.

As we have mentioned, we can use any priority queue to develop a sorting method, as shown in Program 9.6. We insert all the keys to be sorted into the priority queue, then repeatedly use *remove the maximum* to remove them all in decreasing order. Using a priority queue represented as an unordered list in this way corresponds to doing a selection sort; using an ordered list corresponds to doing an insertion sort.

Figures 9.5 and 9.6 give an example of the first phase (the construction process) when a heap-based priority-queue implementation is used; Figures 9.7 and 9.8 show the second phase (which we refer to as the *sortdown* process) for the heap-based implementation. For practical purposes, this method is comparatively inelegant, because it unnecessarily makes an extra copy of the items to be sorted (in the priority queue). Also, using N successive insertions is not the most efficient way to build a heap from N given elements. Next, we address these two points and derive the classical heapsort algorithm.

Exercises

▷ **9.21** Give the heap that results when the keys E A S Y Q U E S T I O N are inserted into an initially empty heap.

▷ **9.22** Using the conventions of Exercise 9.1, give the sequence of heaps produced when the operations

P R I O * R * * I * T * Y * * * Q U E * * * U * E

are performed on an initially empty heap.

9.23 Because the `exch` primitive is used in the heapify operations, the items are loaded and stored twice as often as necessary. Give more efficient implementations that avoid this problem, à la insertion sort.

9.24 Why do we not use a sentinel to avoid the `j<N` test in `sink`?

∘ **9.25** Add the *replace the maximum* operation to the heap-based priority-queue implementation of Program 9.5. Be sure to consider the case when the value to be added is larger than all values in the queue. *Note*: Use of `pq[0]` leads to an elegant solution.

9.26 What is the minimum number of keys that must be moved during a *remove the maximum* operation in a heap? Give a heap of size 15 for which the minimum is achieved.

9.27 What is the minimum number of keys that must be moved during three successive *remove the maximum* operations in a heap? Give a heap of size 15 for which the minimum is achieved.

9.4 Heapsort

We can adapt the basic idea in Program 9.6 to sort an array without needing any extra space, by maintaining the heap within the array to be sorted. That is, focusing on the task of sorting, we abandon the notion of hiding the representation of the priority queue, and rather

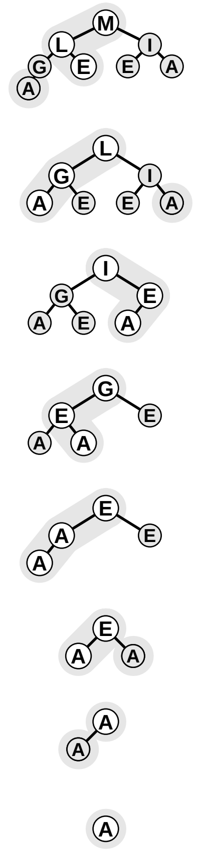


Figure 9.8
Sorting from a heap (continued)

This sequence depicts removal of the rest of the keys from the heap in Figure 9.7. Even if every element goes all the way back to the bottom, the total cost of the sorting phase is less than

$$\lg N + \ldots + \lg 2 + \lg 1,$$

which is less than $N \log N$.

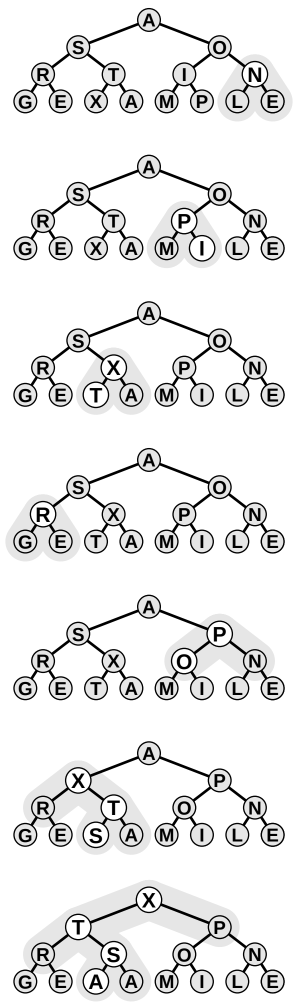


Figure 9.9
Bottom-up heap construction

Working from right to left and bottom to top, we construct a heap by ensuring that the subtree below the current node is heap ordered. The total cost is linear in the worst case, because most nodes are near the bottom.

than being constrained by the interface to the priority-queue ADT, we use `swim` and `sink` directly.

Using Program 9.5 directly in Program 9.6 corresponds to proceeding from left to right through the array, using `swim` to ensure that the elements to the left of the scanning pointer make up a heap-ordered complete tree. Then, during the sortdown process, we put the largest element into the place vacated as the heap shrinks. That is, the sortdown process is like selection sort, but it uses a more efficient way to find the largest element in the unsorted part of the array.

Rather than constructing the heap via successive insertions as shown in Figures 9.5 and 9.6, it is more efficient to build the heap by going backward through it, making little subheaps from the bottom up, as shown in Figure 9.9. That is, we view every position in the array as the root of a small subheap and take advantage of the fact that `sink` works as well for such subheaps as it does for the big heap. If the two children of a node are heaps, then calling `sink` on that node makes the subtree rooted there a heap. By working backward through the heap, calling `sink` on each node, we can establish the heap property inductively. The scan starts halfway back through the array because we can skip the subheaps of size 1.

A full implementation is given in Program 9.7, the classical *heapsort* algorithm. Although the loops in this program seem to do different tasks (the first constructs the heap, and the second destroys the heap for the sortdown), they are both built around the `swim` method, which restores order in a tree that is heap-ordered except possibly at the root, using the array representation of a complete tree. Figure 9.10 illustrates the contents of the array for the example corresponding to Figures 9.7 through 9.9.

Property 9.4 *Bottom-up heap construction takes linear time.*

This fact follows from the observation that most of the heaps processed are small. For example, to build a heap of 127 elements, we process 32 heaps of size 3, 16 heaps of size 7, 8 heaps of size 15, 4 heaps of size 31, 2 heaps of size 63, and 1 heap of size 127, so $32 \cdot 1 + 16 \cdot 2 + 8 \cdot 3 + 4 \cdot 4 + 2 \cdot 5 + 1 \cdot 6 = 120$ promotions (twice as many comparisons) are required in the worst case. For $N = 2^n - 1$, an upper bound on

Program 9.7 Heapsort

This code sorts `a[1]`, ..., `a[N]`, using the `sink` method of Program 9.4 (with `exch` and `less` implementations that exchange and compare, respectively, the items in `a` specified by their parameters). The `for` loop constructs the heap; then, the `while` loop exchanges the largest element (`a[1]`) with `a[N]` and repairs the heap, continuing until the heap is empty. This implementation depends on the first element of the array being at index `1` so that it can treat the array as representing a complete tree and compute implicit indices (see Figure 9.2); it is not difficult to shift indices to implement our standard interface to sort a subarray `a[l]`, ..., `a[r]` (see Exercise 9.30).

```
for (int k = N/2; k >= 1; k--)
  sink(k, N);
while (N > 1)
  { exch(1, N); sink(1, --N); }
```

the number of promotions is

$$\sum_{1 \le k < n} k2^{n-k-1} = 2^n - n - 1 < N.$$

A similar proof holds when $N + 1$ is not a power of 2. ■

This property is not of particular importance for heapsort, because its time is still dominated by the $N \log N$ time for the sortdown, but it is important for other priority-queue applications, where a linear-time *construct* operation can lead to a linear-time algorithm. As noted in Figure 9.6, constructing a heap with N successive *insert* operations requires a total of $N \log N$ steps in the worst case (even though the total turns out to be linear on the average for random files).

Property 9.5 *Heapsort uses fewer than* $2N \lg N$ *comparisons to sort* N *elements.*

The slightly higher bound $3N \lg N$ follows immediately from Property 9.2. The bound given here follows from a more careful count based on Property 9.4. ■

Property 9.5 and the in-place property are the two primary reasons that heapsort is of practical interest: It is *guaranteed* to sort N elements in place in time proportional to $N \log N$, no matter what the

```
A S O R T I N G E X A M P L E
A S O R T I N G E X A M P L E
A S O R T P N G E X A M I L E
A S O R X P N G E T A M I L E
A S O R X P N G E T A M I L E
A S P R X O N G E T A M I L E
A X P R T O N G E S A M I L E
X T P R S O N G E A A M I L E

T S P R E O N G E A A M I L X
S R P L E O N G E A A M I T X
R L P I E O N G E A A M S T X
P L O I E M N G E A A R S T X
O L N I E M A G E A P R S T X
N L M I E A A G E O P R S T X
M L E I E A A G N O P R S T X
L I E G E A A M N O P R S T X
I G E A E A L M N O P R S T X
G E E A A I L M N O P R S T X
E A E A G I L M N O P R S T X
E A A E G I L M N O P R S T X
A A E E G I L M N O P R S T X
A A E E G I L M N O P R S T X
```

Figure 9.10
Heapsort example

Heapsort is an efficient selection-based algorithm. First, we build a heap from the bottom up, in-place. The top eight lines in this figure correspond to Figure 9.9. Next, we repeatedly remove the largest element in the heap. The unshaded parts of the bottom lines correspond to Figures 9.7 and 9.8; the shaded parts contain the growing sorted file.

input. There is no worst-case input that makes heapsort run significantly slower (unlike quicksort), and heapsort does not use any extra space (unlike mergesort). This guaranteed worst-case performance does come at a price: for example, the algorithm's inner loop (cost per comparison) has more basic operations than quicksort's, and it uses more comparisons than quicksort for random files, so heapsort is likely to be slower than quicksort for typical or random files.

Heaps are also useful for solving the *selection* problem of finding the k largest of N items (see Chapter 7), particularly if k is small. We simply stop the heapsort algorithm after k items have been taken from the top of the heap.

Property 9.6 *Heap-based selection allows the kth largest of N items to be found in time proportional to N when k is small or close to N, and in time proportional to $N \log N$ otherwise.*

One option is to build a heap, using fewer than $2N$ comparisons (by Property 9.4), then to remove the k largest elements, using $2k \lg N$ or fewer comparisons (by Property 9.2), for a total of $2N + 2k \lg N$. Another method is to build a minimum-oriented heap of size k, then to perform k *replace the minimum* (*insert* followed by *remove the minimum*) operations with the remaining elements for a total of at most $2k + 2(N - k) \lg k$ comparisons (see Exercise 9.36). This method uses space proportional to k, so is attractive for finding the k largest of N elements when k is small and N is large (or is not known in advance). For random keys and other typical situations, the $\lg k$ upper bound for heap operations in the second method is likely to be $O(1)$ when k is small relative to N (see Exercise 9.37). ■

Various ways to improve heapsort further have been investigated. One idea, developed by Floyd, is to note that an element reinserted into the heap during the sortdown process usually goes all the way to the bottom. Thus, we can save time by avoiding the check for whether the element has reached its position, simply promoting the larger of the two children until the bottom is reached, then moving back up the heap to the proper position. This idea cuts the number of comparisons by a factor of 2 asymptotically—close to the $\lg N! \approx N \lg N - N/\ln 2$ that is the absolute minimum number of comparisons needed by any sorting algorithm (see Part 8). The method requires extra bookkeeping, and it is useful in practice only when the cost of comparisons is relatively high (for example, when we are sorting records with strings or other types of long keys).

Another idea is to build heaps based on an array representation of complete heap-ordered *ternary* trees, with a node at position k larger than or equal to nodes at positions $3k-1$, $3k$, and $3k+1$ and smaller than or equal to nodes at position $\lfloor (k+1)/3 \rfloor$, for positions between 1 and N in an array of N elements. There is a tradeoff between the lower cost from the reduced tree height and the higher cost of finding the largest of the three children at each node. This tradeoff is dependent on details of the implementation (see Exercise 9.31) and on the expected relative frequency of *insert*, *remove the maximum*, and *change priority* operations.

Figure 9.11 shows heapsort in operation on a randomly ordered file. At first, the process seems to do anything but sort, because large elements are moving to the beginning of the file as the heap is being constructed. But then the method looks more like a mirror image of selection sort, as expected. Figure 9.12 shows that different types of input files can yield heaps with peculiar characteristics, but they look more like random heaps as the sort progresses.

Naturally, we are interested in how to choose among heapsort, quicksort, and mergesort for a particular application. The choice between heapsort and mergesort essentially reduces to a choice between a sort that is not stable (see Exercise 9.28) and one that uses extra memory; the choice between heapsort and quicksort reduces to a choice between average-case speed and worst-case speed. Having dealt extensively with improving the inner loops of quicksort and mergesort, we leave this activity for heapsort as exercises in this chapter. Making heapsort faster than quicksort is typically not in the cards—as indicated by the empirical studies in Table 9.2—but people interested in fast sorts on their machines will find the exercise instructive. As usual, various specific properties of machines and programming environments can play an important role. For example, quicksort and mergesort have a locality property that gives them a further advantage on certain machines. When comparisons are extremely expensive, Floyd's version is the method of choice, as it is nearly optimal in terms of time and space costs in such situations.

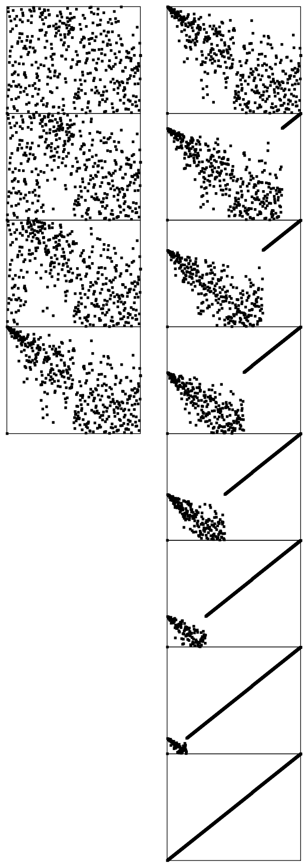


Figure 9.11
Dynamic characteristics of heapsort

The construction process (left) *seems to unsort the file, putting large elements near the beginning. Then, the sortdown process* (right) *works like selection sort, keeping a heap at the beginning and building up the sorted array at the end of the file.*

Exercises

9.28 Show that heapsort is not stable.

Figure 9.12
Dynamic characteristics of heapsort on various types of files

The running time for heapsort is not particularly sensitive to the input. No matter what the input values are, the largest element is always found in less than lg N *steps. These diagrams show files that are random, Gaussian, nearly ordered, nearly reverse-ordered, and randomly ordered with 10 distinct key values* (at the top, left to right). *The second diagrams from the top show the heap constructed by the bottom-up algorithm, and the remaining diagrams show the sortdown process for each file. The heaps somewhat mirror the initial file at the beginning, but all become more like the heaps for a random file as the process continues.*

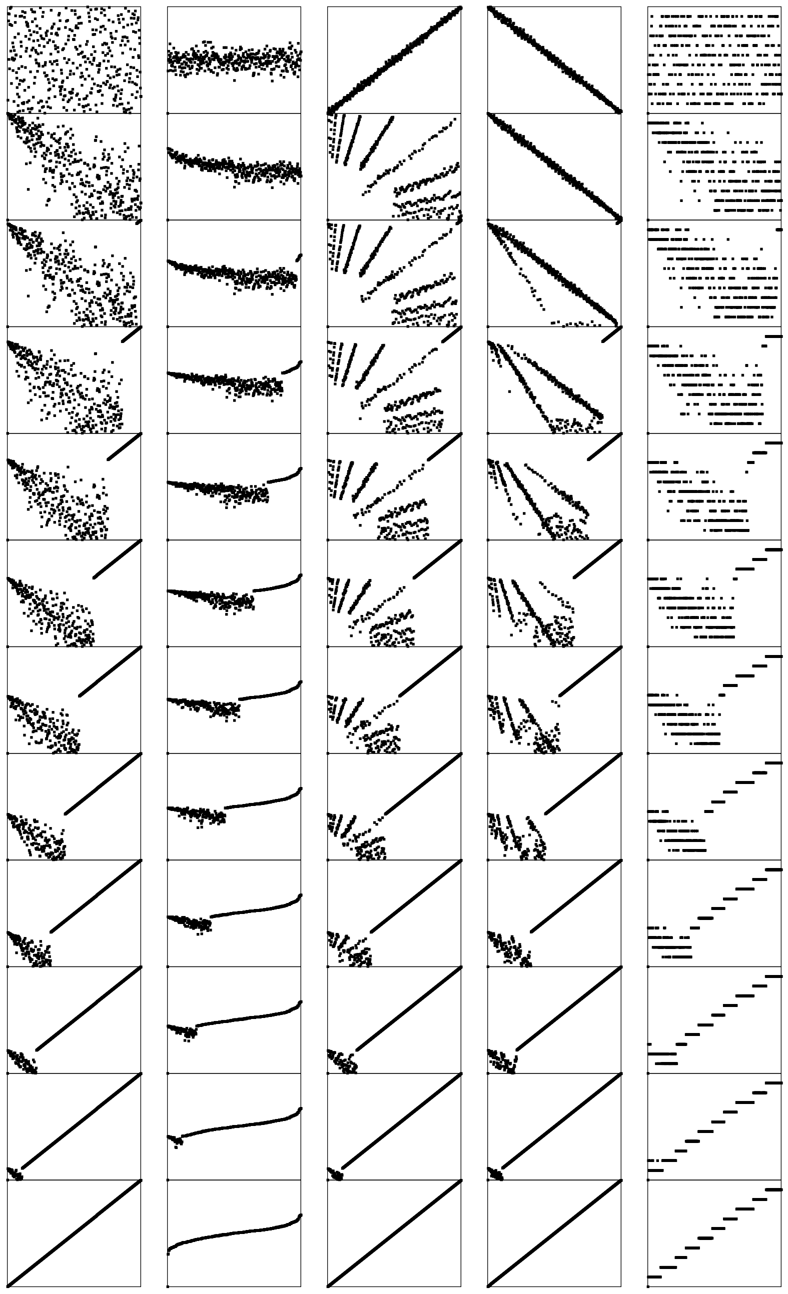


Table 9.2 Empirical study of heapsort algorithms

The relative timings for various sorts on files of random integers in the left part of the table confirm our expectations from the lengths of the inner loops that heapsort is slower than quicksort but competitive with mergesort. The timings for the first N words of *Moby Dick* in the right part of the table show that Floyd's method is an effective improvement to heapsort when comparisons are expensive.

	32-bit integer keys					*string keys*		
N	Q	M	PQ	H	F	Q	H	F
12500	22	21	53	23	24	106	141	109
25000	20	32	75	34	34	220	291	249
50000	42	70	156	71	67	518	756	663
100000	95	152	347	156	147	1141	1797	1584
200000	194	330	732	352	328			
400000	427	708	1690	818	768			
800000	913	1524	3626	1955	1851			

Key:

Q	Quicksort with cutoff for small files
M	Mergesort with cutoff for small files
PQ	Priority-queue–based heapsort (Program 9.6)
H	Heapsort, standard implementation (Program 9.7)
F	Heapsort with Floyd's improvement

• **9.29** Empirically determine the percentage of time heapsort spends in the construction phase for $N = 10^3$, 10^4, 10^5, and 10^6.

9.30 Write a heapsort-based implementation of our standard `sort` method, which sorts the subarray `a[l]`, ... , `a[r]`.

• **9.31** Implement a version of heapsort based on complete heap-ordered ternary trees, as described in the text. Compare the number of comparisons used by your program empirically with the standard implementation, for $N = 10^3$, 10^4, 10^5, and 10^6.

• **9.32** Continuing Exercise 9.31, determine empirically whether or not Floyd's method is effective for ternary heaps.

○ **9.33** Considering the cost of comparisons only, and assuming that it takes t comparisons to find the largest of t elements, find the value of t that minimizes the coefficient of $N \log N$ in the comparison count when a t-ary heap is used in heapsort. First, assume a straightforward generalization of Program 9.7; then, assume that Floyd's method can save one comparison in the inner loop.

∘ **9.34** For $N = 32$, give an arrangement of keys that makes heapsort use as many comparisons as possible.

•• **9.35** For $N = 32$, give an arrangement of keys that makes heapsort use as few comparisons as possible.

9.36 Prove that building a priority queue of size k then doing $N - k$ *replace the minimum* (*insert* followed by *remove the minimum*) operations leaves the k largest of the N elements in the heap.

9.37 Implement both of the versions of heapsort-based selection referred to in the discussion of Property 9.6, using the method described in Exercise 9.25. Compare the number of comparisons they use empirically with the quicksort-based method from Chapter 7, for $N = 10^6$ and $k = 10, 100, 1000, 10^4, 10^5$, and 10^6.

• **9.38** Implement a version of heapsort based on the idea of representing the heap-ordered tree in preorder rather than in level order. Empirically compare the number of comparisons used by this version with the number used by the standard implementation, for randomly ordered keys with $N = 10^3, 10^4, 10^5$, and 10^6.

9.5 Priority-Queue ADT

For most applications of priority queues, we want to arrange to have the priority-queue method, instead of returning values for *remove the maximum*, tell us *which* of the records has the largest key, and to work in a similar fashion for the other operations. That is, we assign priorities and use priority queues for only the purpose of accessing other information in an appropriate order. This arrangement is akin to use of the *indirect-sort* or the *pointer-sort* concepts described in Chapter 6. In particular, this approach is required for operations such as *change priority* or *remove* to make sense. We examine an implementation of this idea in detail here, both because we shall be using priority queues in this way later in the book and because this situation is prototypical of the problems we face when we design interfaces and implementations for ADTs.

When we want to *remove* an item from a priority queue, how do we specify which item? When we want to maintain multiple priority queues, how do we organize the implementations so that we can manipulate priority queues in the same way that we manipulate other

Program 9.8 Full priority-queue ADT

This interface for a priority-queue ADT allows client programs to delete items and to change priorities (using object handles provided by the implementation) and to merge priority queues together.

```
class PQfull // ADT interface
  { // implementations and private members hidden
    boolean empty()
    Object insert(ITEM)
    ITEM getmax()
    void change(Object, ITEM)
    void remove(Object)
    void join(PQfull)
  };
```

types of data? Questions such as these are the topic of Chapter 4. Program 9.8 gives a general interface for priority queues along the lines that we discussed in Section 4.9. It supports a situation where a client has keys and associated information and, while primarily interested in the operation of accessing the information associated with the highest key, may have numerous other data-processing operations to perform on the objects, as we discussed at the beginning of this chapter. All operations refer to a particular priority queue through a handle (a pointer to an object whose class is not specified). The *insert* operation returns a handle for each object added to the priority queue by the client program. In this arrangement, client programs are responsible for keeping track of handles, which they may later use to specify which objects are to be affected by *remove* and *change priority* operations, and which priority queues are to be affected by all of the operations.

This arrangement places restrictions on both the client and the implementation. The client is not given a way to access information through handles except through this interface. It has the responsibility to use the handles properly: for example, there is no good way for an implementation to check for an illegal action such as a client using a handle to an item that is already removed. For its part, the implementation cannot move around information freely, because clients have handles that they may use later. This point will become clearer when we examine details of implementations. As usual, whatever level of detail we choose in our implementations, an abstract interface such as Program 9.8 is a useful starting point for making tradeoffs between the needs of applications and the needs of implementations.

Implementations of the basic priority-queue operations, using an unordered doubly linked-list representation, are given in Programs 9.9 and 9.10. Most of the code is based on elementary linked list operations from Section 3.3, but the implementation of the client handle abstraction is noteworthy in the present context: the `insert` method returns an `Object`, which the client can only take to mean "a reference to an object of some unspecified class" since `Node`, the actual type of the object, is private. So the client can do little else with the reference but keep it in some data structure associated with the item that it provided as a parameter to `insert`. But if the client needs to *change priority* the item's key or to *remove* the item from the priority queue, this object is precisely what the implementation needs to be able to accomplish the task: the appropriate methods can cast the type to `Node` and make the necessary modifications. It is easy to develop other, similarly straightforward, implementations using other elementary representations (see, for example, Exercise 9.40).

As we discussed in Section 9.1, the implementation given in Programs 9.9 and 9.10 is suitable for applications where the priority queue is small and *remove the maximum* or *find the maximum* operations are infrequent; otherwise, heap-based implementations are preferable. Implementing `swim` and `sink` for heap-ordered trees with explicit links while maintaining the integrity of the handles is a challenge that we leave for exercises, because we shall be considering two alternative approaches in detail in Sections 9.6 and 9.7.

A full ADT such as Program 9.8 has many virtues, but it is sometimes advantageous to consider other arrangements, with different restrictions on the client programs and on implementations. In Section 9.6 we consider an example where the client program keeps the responsibility for maintaining the records and keys, and the priority-queue routines refer to them indirectly.

Slight changes in the interface also might be appropriate. For example, we might want a method that returns the value of the highest priority key in the queue, rather than just a way to reference that key and its associated information. Also, the issues that we considered in Sections 4.9 and 4.10 associated with memory management and

Program 9.9 Unordered doubly linked list priority queue

This implementation includes the *construct*, *test if empty*, and *insert* methods from the interface of Program 9.8 (see Program 9.10 for implementations of the other four methods). It maintains a simple unordered list, with head and tail nodes. We specify the class `Node` to be a doubly linked list node (with an item and two links). The private data fields are just the list's head and tail links.

```
class PQfull
  {
    private static class Node
      { ITEM key; Node prev, next;
        Node(ITEM v)
          { key = v; prev = null; next = null; }
      }
    private Node head, tail;
    PQfull()
      {
        head = new Node(null);
        tail = new Node(null);
        head.prev = tail; head.next = tail;
        tail.prev = head; tail.next = head;
      }
    boolean empty()
      { return head.next.next == head; }
    Object insert(ITEM v)
      { Node t = new Node(v);
        t.next = head.next; t.next.prev = t;
        t.prev = head; head.next = t;
        return t;
      }
    ITEM getmax()
      // See Program 9.10
    void change(Object x, ITEM v)
      // See Program 9.10
    void remove(Object x)
      // See Program 9.10
    void join(PQfull p)
      // See Program 9.10
  }
```

copy semantics come into play. We are not considering the *copy* operation and have chosen just one out of several possibilities for *join* (see Exercises 9.44 and 9.45).

It is easy to add such procedures to the interface in Program 9.8, but it is much more challenging to develop an implementation where logarithmic performance for all operations is guaranteed. In applications where the priority queue does not grow to be large, or where the mix of *insert* and *remove the maximum* operations has some special properties, a fully flexible interface might be desirable. But in applications where the queue will grow to be large, and where a tenfold or a hundredfold increase in performance might be noticed or appreciated, it might be worthwhile to restrict to the set of operations where efficient performance is assured. A great deal of research has gone into the design of priority-queue algorithms for different mixes of operations; the binomial queue described in Section 9.7 is an important example.

Exercises

9.39 Which priority-queue implementation would you use to find the 100 smallest of a set of 10^6 random numbers? Justify your answer.

9.40 Provide implementations similar to Programs 9.9 and 9.10 that use *ordered* doubly linked lists. *Note*: Because the client has handles into the data structure, your programs can change only links (rather than keys) in nodes.

9.41 Provide implementations for *insert* and *remove the maximum* (the priority-queue interface in Program 9.1) using complete heap-ordered trees represented with explicit nodes and links. *Note*: Because the client has no handles into the data structure, you can take advantage of the fact that it is easier to exchange information fields in nodes than to exchange the nodes themselves.

• **9.42** Provide implementations for *insert*, *remove the maximum*, *change priority*, and *remove* (the priority-queue interface in Program 9.8) using heap-ordered trees with explicit links. *Note*: Because the client has handles into the data structure, this exercise is more difficult than Exercise 9.41, not just because the nodes have to be triply linked, but also because your programs can change only links (rather than keys) in nodes.

9.43 Add a (brute-force) implementation of the *join* operation to your implementation from Exercise 9.42.

∘ **9.44** Suppose that we add a `clone` method to Program 9.8 (and specify that every implementation `implements Cloneable`). Add an implementation of `clone` to Programs 9.9 and 9.10, and write a driver program that tests your interface and implementation.

Program 9.10 Doubly linked list priority queue (continued)

These method implementations complete the priority queue implementation of Program 9.9. The *remove the maximum* operation requires scanning through the whole list, but the overhead of maintaining doubly linked lists is justified by the fact that the *change priority*, *remove*, and *join* operations all are implemented in constant time, using only elementary operations on the lists (see Chapter 3 for more details on doubly linked lists).

The `change` and `remove` methods take an `Object` reference as a parameter, which must reference an object of (private) type `Node`—a client can only get such a reference from `insert`.

We might make this class `Cloneable` and implement a `clone` method that makes a copy of the whole list (see Section 4.9), but client object handles would be invalid for the copy. The `join` implementation appropriates the list nodes from the parameter to be included in the result, but it does *not* make copies of them, so client handles remain valid.

```
ITEM getmax()
  { ITEM max; Node x = head.next;
    for (Node t = x; t.next != head; t = t.next)
      if (Sort.less(x.key, t.key)) x = t;
    max = x.key;
    remove(x);
    return max;
  }
void change(Object x, ITEM v)
  { ((Node) x).key = v; }
void remove(Object x)
  { Node t = (Node) x;
    t.next.prev = t.prev;
    t.prev.next = t.next;
  }
void join(PQfull p)
  {
    tail.prev.next = p.head.next;
    p.head.next.prev = tail.prev;
    head.prev = p.tail;
    p.tail.next = head;
    tail = p.tail;
  }
```

• **9.45** Change the interface and implementation for the *join* operation in Programs 9.9 and 9.10 such that it returns a `PQfull` (the result of joining the parameters).

9.46 Provide a priority-queue interface and implementation that supports *construct* and *remove the maximum*, using tournaments (see Section 5.7). Program 5.19 will provide you with the basis for *construct*.

• **9.47** Add *insert* to your solution to Exercise 9.46.

9.6 Priority Queues for Client Arrays

Suppose that the records to be processed in a priority queue are in an existing array. In this case, it makes sense to have the priority-queue routines refer to items through the array index. Moreover, we can use the array index as a handle to implement all the priority-queue operations. An interface along these lines is illustrated in Program 9.11. Figure 9.13 shows how this approach might apply for a small example. Without copying or making special modifications of records, we can keep a priority queue containing a subset of the records.

Using indices into an existing array is a natural arrangement, but it leads to implementations with an orientation opposite to that of Program 9.8. Now it is the client program that cannot move around information freely, because the priority-queue routine is maintaining indices into data maintained by the client. For its part, the priority queue implementation must not use indices without first being given them by the client.

To develop an implementation, we use precisely the same approach as we did for index sorting in Section 6.8. We manipulate indices and define `less` such that comparisons reference the client's array. There are added complications here, because the priority-queue routine must keep track of the objects so that it can find them when the client program refers to them by the handle (array index). To this end, we add a second index array to keep track of the position of the keys in the priority queue. To localize the maintenance of this array, we move data only with the `exch` operation and define `exch` appropriately.

Program 9.12 is a heap-based implementation of this approach that differs only slightly from Program 9.5 but is well worth studying because it is so useful in practice. We refer to the data structure built by this program as an *index heap*. We shall use this program as a

Program 9.11 Priority-queue ADT interface for index items

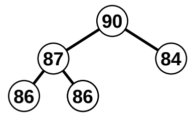
Instead of building a data structure from the items themselves, this interface provides for building a priority queue using indices into a client array. The constructor takes a reference to an array as a parameter, and the *insert*, *remove the maximum*, *change priority*, and *remove* methods all use indices into that array and compare array entries with `ITEM`'s `less` method. For example, the client program might define `less` so that `less(i, j)` is the result of comparing `data[i].grade` and `data[j].grade`.

```
class PQi // ADT interface
  { // implementations and private members hidden
    PQi(Array)
    boolean empty()
    void insert(int)
    int getmax()
    void change(int)
    void remove(int)
  };
```

building block for other algorithms in Parts 5 through 7. As usual, we do no error checking, and we assume (for example) that indices are always in the proper range and that the user does not try to insert anything on a full queue or to remove anything from an empty one. Adding code for such checks is straightforward.

We can use the same approach for any priority queue that uses an array representation (for example, see Exercises 9.50 and 9.51). The main disadvantage of using indirection in this way is the extra space used. The size of the index arrays has to be the size of the data array, when the maximum size of the priority queue could be much less.

Other approaches to building a priority queue on top of existing data in an array are either to have the client program make records consisting of a key with its array index as associated information or to use a class for index keys with its own `less` method. Then, if the implementation uses a linked-allocation representation such as the one in Programs 9.9 and 9.10 or Exercise 9.42, then the space used by the priority queue would be proportional to the maximum number of elements on the queue at any one time. Such approaches would



k	qp[k]	pq[k]	data[k]	
0			Wilson	63
1	5	3	Johnson	86
2	2	2	Jones	87
3	1	4	Smith	90
4	3	9	Washington	84
5		1	Thompson	65
6			Brown	82
7			Jackson	61
8			White	76
9	4		Adams	86
10			Black	71

Figure 9.13
Index heap data structures

By manipulating indices, rather than the records themselves, we can build a priority queue on a subset of the records in an array. Here, a heap of size 5 in the array `pq` *contains the indices to those students with the top five grades. Thus,* `data[pq[1]].name` *contains* Smith*, the name of the student with the highest grade, and so forth. An inverse array* `qp` *allows the priority-queue routines to treat the array indices as handles. For example, if we need to change* Smith*'s grade to* 85*, we change the entry in* `data[3].grade`*, then call* `PQchange(3)`*. The priority-queue implementation accesses the record at* `pq[qp[3]]` *(or* `pq[1]`*, because* `qp[3]=1`*) and the new key at* `data[pq[1]].name` *(or* `data[3].name`*, because* `pq[1]=3`*).*

Program 9.12 Index-heap–based priority queue

This implementation of Program 9.11 maintains pq as an array of indices into a client array.

We keep the heap position corresponding to index value k in qp[k], which allows us to implement *change priority* (see Figure 9.14) and *remove* (see Exercise 9.49). We maintain the invariant pq[qp[k]]=qp[pq[k]]=k for all k in the heap (see Figure 9.13). The methods less and exch are the key to the implementation—they allow us to use the same sink and swim code as for standard heaps.

```
class PQi
  {
    private boolean less(int i, int j)
      { return a[pq[i]].less(pq[j]); }
    private void exch(int i, int j)
      { int t = pq[i]; pq[i] = pq[j]; pq[j] = t;
        qp[pq[i]] = i; qp[pq[j]] = j;
      }
    private void swim(int k)
      // Program 9.3
    private void sink(int k, int N)
      // Program 9.4
    private ITEM[] a;
    private int[] pq, qp;
    private int N;
    PQi(ITEM[] items)
      { a = items; N = 0;
        pq = new int[a.length+1];
        qp = new int[a.length+1];
      }
    boolean empty()
      { return N == 0; }
    void insert(int v)
      { pq[++N] = v; qp[v] = N; swim(N); }
    int getmax()
      { exch(1, N); sink(1, N-1); return pq[N--]; }
    void change(int k)
      { swim(qp[k]); sink(qp[k], N); }
  };
```

be preferred over Program 9.12 if space must be conserved and if the priority queue involves only a small fraction of the data array.

Contrasting this approach to providing a complete priority-queue implementation to the approach in Section 9.5 exposes essential differences in ADT design. In the first case (Programs 9.9 and 9.10, for example), it is the responsibility of the priority-queue implementation to allocate and deallocate the memory for the keys, to change key values, and so forth. The ADT supplies the client with handles to items, and the client accesses items only through calls to the priority-queue routines, using the handles as parameters. In the second case (Program 9.12, for example), the client is responsible for the keys and records, and the priority-queue routines access this information only through handles provided by the user (array indices, in the case of Program 9.12). Both uses require cooperation between client and implementation.

Note that, in this book, we are normally interested in cooperation beyond that encouraged by programming language support mechanisms. In particular, we want the performance characteristics of the implementation to match the dynamic mix of operations required by the client. One way to ensure that match is to seek implementations with provable worst-case performance bounds, but we can solve many problems more easily by matching their performance requirements with simpler implementations.

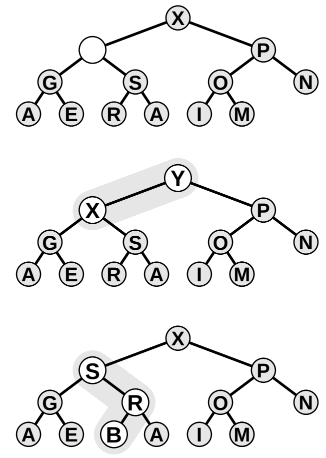


Figure 9.14
Changing of the priority of a node in a heap

The top diagram depicts a heap that is heap-ordered, except possibly at one given node. If the node is larger than its parent, then it must move up, just as depicted in Figure 9.3. This situation is illustrated in the middle diagram, with Y *moving up the tree (in general, it might stop before hitting the root). If the node is smaller than the larger of its two children, then it must move down, just as depicted in Figure 9.4. This situation is illustrated in the bottom diagram, with* B *moving down the tree (in general, it might stop before hitting the bottom). We can use this procedure in two ways: as the basis for the* change priority *operation on heaps in order to reestablish the heap condition after changing the key in a node, or as the basis for the remove operation on heaps in order to reestablish the heap condition after replacing the key in a node with the rightmost key on the bottom level.*

Exercises

9.48 Suppose that an array is filled with the keys E A S Y Q U E S T I O N. Give the contents of the `pq` and `qp` arrays after these keys are inserted into an initially empty heap using Program 9.12.

∘ 9.49 Add a *remove* operation to Program 9.12.

9.50 Implement the priority-queue ADT for index items (see Program 9.11) using an ordered-array representation for the priority queue.

9.51 Implement the priority-queue ADT for index items (see Program 9.11) using an unordered-array representation for the priority queue.

∘ 9.52 Given an array `a` of N elements, consider a complete binary tree of $2N$ elements (represented as an array `pq`) containing indices from the array with the following properties: (*i*) for `i` from `0` to `N-1`, we have `pq[N+i]=i`; and (*ii*) for `i` from `1` to `N-1`, we have `pq[i]=pq[2*i]` if `a[pq[2*i]]>a[pq[2*i+1]]`, and we have `pq[i]=pq[2*i+1]` otherwise. Such a structure is called an *index heap tournament* because it combines the features of index heaps and tournaments (see Program 5.19). Give the index heap tournament corresponding to the keys E A S Y Q U E S T I O N.

∘ **9.53** Implement the priority-queue ADT for index items (see Program 9.11) using an index heap tournament (see Exercise 9.46).

9.7 Binomial Queues

None of the implementations that we have considered admit implementations of *join*, *remove the maximum*, and *insert* that are all efficient in the worst case. Unordered linked lists have fast *join* and *insert*, but slow *remove the maximum*; ordered linked lists have fast *remove the maximum*, but slow *join* and *insert*; heaps have fast *insert* and *remove the maximum*, but slow *join*; and so forth. (See Table 9.1.) In applications where frequent or large *join* operations play an important role, we need to consider more advanced data structures.

In this context, we mean by "efficient" that the operations should use no more than logarithmic time in the worst case. This restriction would seem to rule out array representations, because we can join two large arrays apparently only by moving all the elements in at least one of them. The unordered doubly linked-list representation of Program 9.9 does the *join* in constant time but requires that we walk through the whole list for *remove the maximum*. Use of a doubly linked ordered list (see Exercise 9.40) gives a constant-time *remove the maximum* but requires linear time to merge lists for *join*.

Numerous data structures have been developed that can support efficient implementations of all the priority-queue operations. Most of them are based on direct linked representation of heap-ordered trees. Two links are needed for moving down the tree (either to both children in a binary tree or to the first child and next sibling in a binary tree representation of a general tree), and one link to the parent is needed for moving up the tree. Developing implementations of the heap-ordering operations that work for any (heap-ordered) tree shape with explicit nodes and links or other representation is generally straightforward. The difficulty lies in dynamic operations such as *insert*, *remove*, and *join*, which require us to modify the tree structure. Different data structures are based on different strategies for modifying the tree structure while still maintaining balance in the tree. The algorithms use trees that are both more flexible than are complete trees and keep the trees sufficiently balanced to ensure a logarithmic time bound.

The overhead of maintaining a triply linked structure—ensuring that a particular implementation correctly maintains three pointers in all circumstances—can be burdensome (see Exercise 9.41). Moreover, in many practical situations, it is difficult to demonstrate that efficient implementations of *all* the operations are required, so we might pause before taking on such an implementation. On the other hand, it is also difficult to demonstrate that efficient implementations are *not* required, and the investment to guarantee that all the priority-queue operations will be fast may be justified. Regardless of such considerations, the next step from heaps to a data structure that allows for efficient implementation of *join*, *insert*, and *remove the maximum* is fascinating and worthy of study in its own right.

Even with a linked representation for the trees, the heap condition and the condition that the heap-ordered binary tree be complete are too strong to allow efficient implementation of the *join* operation. Given two heap-ordered trees, how do we merge them together into just one tree? For example, if one of the trees has 1023 nodes and the other has 255 nodes, how can we merge them into a tree with 1278 nodes, without touching more than 10 or 20 nodes? It seems impossible to merge heap-ordered trees in general if the trees are to be heap ordered and complete, but various advanced data structures have been devised that weaken the heap-order and balance conditions to get the flexibility that we need to devise an efficient *join*. Next, we consider an ingenious solution to this problem, called the *binomial queue*, that was developed by Vuillemin in 1978.

To begin, we note that the *join* operation is trivial for one particular type of tree with a relaxed heap-ordering restriction.

Definition 9.4 *A binary tree comprising nodes with keys is said to be* **left-heap-ordered** *if the key in each node is larger than or equal to all the keys in that node's left subtree (if any).*

Definition 9.5 *A* **power-of-2 heap** *is a left-heap-ordered tree consisting of a root node with an empty right subtree and a complete left subtree. The tree corresponding to a power-of-2 heap by the left-child, right-sibling correspondence is called a* **binomial tree**.

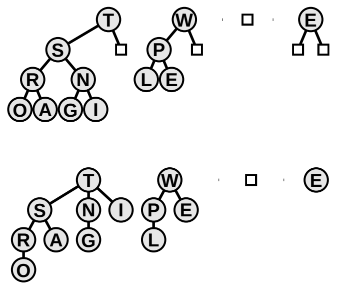


Figure 9.15
A binomial queue of size 13

A binomial queue of size N is a list of left-heap-ordered power-of-2 heaps, one for each bit in the binary representation of N. Thus, a binomial queue of size $13 = 1101_2$ consists of an 8-heap, a 4-heap, and a 1-heap. Shown here are the left-heap-ordered power-of-2 heap representation (top) and the heap-ordered binomial-tree representation (bottom) of the same binomial queue.

Program 9.13 Joining of two equal-sized power-of-2 heaps

We need to change only a few links to combine two equal-sized power-of-2 heaps into one power-of-2 heap that is twice that size. This method, which we define as a private method in the implementation, is one key to the efficiency of the binomial queue algorithm.

```
static Node pair(Node p, Node q)
  {
    if (p.item.less(q.item))
      { p.r = q.l; q.l = p; return q; }
    else { q.r = p.l; p.l = q; return p; }
  }
```

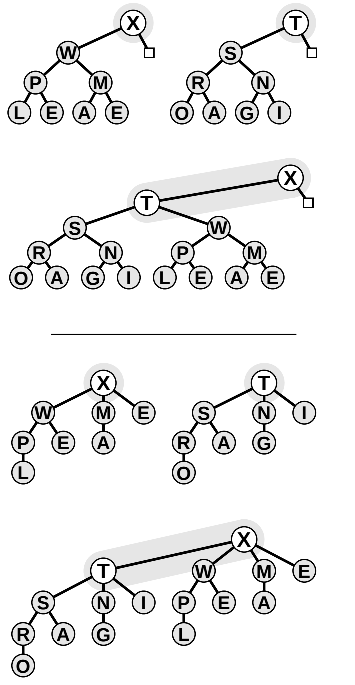


Figure 9.16
Joining of two equal-sized power-of-2 heaps.

We join two power-of-two heaps (top) *by putting the larger of the roots at the root, with that root's* (left) *subtree as the right subtree of the other original root. If the operands have 2^n nodes, the result has 2^{n+1} nodes. If the operands are left-heap-ordered, then so is the result, with the largest key at the root. The heap-ordered binomial-tree representation of the same operation is shown below the line.*

Binomial trees and power-of-2 heaps are equivalent. We work with both representations because binomial trees are slightly easier to visualize, whereas the simple representation of power-of-2 heaps leads to simpler implementations. In particular, we depend upon the following facts, which are direct consequences of the definitions:

- The number of nodes in a power-of-2 heap is a power of 2.
- No node has a key larger than the key at the root.
- Binomial trees are heap-ordered.

The trivial operation upon which binomial-queue algorithms are based is that of joining two power-of-2 heaps that have an equal number of nodes. The result is a heap with twice as many nodes which is easy to create, as illustrated in Figure 9.16. The root node with the larger key becomes the root of the result (with the other original root as the result root's left child), with its left subtree becoming the right subtree of the other root node. Given a linked representation for the trees, the join is a constant-time operation: We simply adjust two links at the top. An implementation is given in Program 9.13. This basic operation is at the heart of Vuillemin's general solution to the problem of implementing priority queues with no slow operations.

Definition 9.6 *A* **binomial queue** *is a set of power-of-2 heaps, no two of the same size. The structure of a binomial queue is determined by that queue's number of nodes, by correspondence with the binary representation of integers.*

A binomial queue of N elements has one power-of-2 heap for each 1 bit in the binary representation of N. For example, a binomial queue of 13 nodes comprises an 8-heap, a 4-heap, and a 1-heap, as illustrated in Figure 9.15. There are at most $\lg N$ power-of-2 heaps in a binomial queue of size N, all of height no greater than $\lg N$.

In accordance with Definitions 9.5 and 9.6, we represent power-of-2 heaps (and handles to items) as links to nodes containing keys and two links (like the explicit tree representation of tournaments in Figure 5.10); and we represent binomial queues as arrays of power-of-2 heaps, by including the following private members in our implementation of Program 9.8:

```
private static class Node
  { ITEM item; Node l, r;
    Node(ITEM v)
      { item = v; l = null; r = null; }
  }
private Node[] bq;
```

The arrays are not large and the trees are not high; and this representation is sufficiently flexible to allow implementation of all the priority-queue operations in less than $\lg N$ steps, as we shall now see.

Each link in the array `bq` is a link to a power-of-2 heap: `bq[i]` is either null or a link to a 2^i-heap. As the queue grows and shrinks, the length of the array increases and decreases, but much less frequently: for example, the array length increases by 1 only after the queue size doubles. It will be convenient to always have the last link null, so we begin with an array of size 1, a null link:

```
PQfull()
  { bq = new Node[1]; bq[0] = null; }
```

thus adopting the convention that a queue is empty if and only if the array is of length 1.

Now, let us consider the *insert* operation. The process of inserting a new item into a binomial queue mirrors precisely the process of incrementing a binary number. To increment a binary number, we move from right to left, changing 1s to 0s because of the carry associated with $1 + 1 = 10_2$, until finding the rightmost 0, which we change to 1. In the analogous way, to add a new item to a binomial queue, we move from right to left, merging heaps corresponding to 1 bits with a

Program 9.14 Insertion into a binomial queue

To insert a node into a binomial queue, we first make the node into a 1-heap, identify it as a carry 1-heap, and then iterate the following process starting at `i = 0`. If the binomial queue has no 2^i-heap, we put the carry 2^i-heap into the queue. If the binomial queue has a 2^i-heap, we combine that with the new one (using the `pair` method from Program 9.13) to make a 2^{i+1}-heap, increment `i`, and iterate until finding an empty heap position in the binomial queue. When we carry in to the null link at the end of the array, we call `grow` to increase the size of the array by 1 and put a null link in the new position (*see text*).

```
Object insert(ITEM v)
  { Node t = new Node(v), c = t;
    for (int i = 0; i < bq.length+1; i++)
      {
        if (c == null) break;
        if (i == bq.length-1) bq = grow(bq);
        if (bq[i] == null) { bq[i] = c; break; }
        c = pair(c, bq[i]); bq[i] = null;
      }
    return t;
  }
```

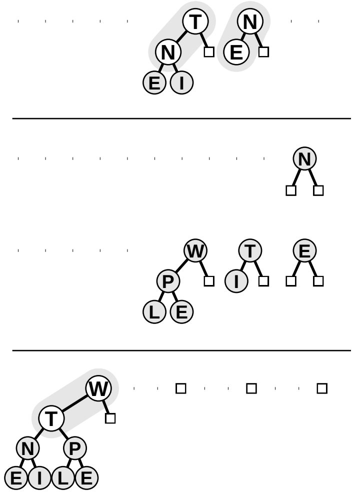


Figure 9.17
Insertion of a new element into a binomial queue

Adding an element to a binomial queue of 7 nodes is analogous to performing the binary addition $111_2 + 1 = 1000_2$*, with carries at each bit. The result is the binomial queue at the bottom, with an 8-heap and null 4-, 2-, and 1-heaps.*

carry heap, until finding the rightmost empty position to put the carry heap.

Specifically, to insert a new item into a binomial queue, we make the new item into a 1-heap. Then, if N is even (rightmost bit 0), we just put this 1-heap in the empty rightmost position of the binomial queue. If N is odd (rightmost bit 1), we join the 1-heap corresponding to the new item with the 1-heap in the rightmost position of the binomial queue to make a carry 2-heap. If the position corresponding to 2 in the binomial queue is empty, we put the carry heap there; otherwise, we merge the carry 2-heap with the 2-heap from the binomial queue to make a carry 4-heap, and so forth, continuing until we get to an empty position in the binomial queue. This process is depicted in Figure 9.17; Program 9.14 is an implementation.

When we add an element to a binomial queue with $2^k - 1$ items to make one with 2^k items, we carry into the null link at the end of the array, replacing it with a binomial tree of size 2^k (the rest of the

Program 9.15 Removal of the maximum in a binomial queue

We first scan the root nodes to find the maximum and remove the power-of-2 heap containing the maximum from the binomial queue. We then remove the root node containing the maximum from its power-of-2 heap and temporarily build a binomial queue that contains the remaining constituent parts of the power-of-2 heap. Finally, we use the *join* operation to merge this binomial queue back into the original binomial queue.

```
ITEM getmax()
  { int i, max; ITEM v = null;
    for (i = 0, max = -1; i < bq.length; i++)
      if (bq[i] != null)
        if ((max == -1) || v.less(bq[i].item))
          { max = i; v = bq[max].item; }
    Node[] temp = new Node[max+1]; temp[max] = null;
    Node x = bq[max].l; bq[max] = null;
    for (i = max-1; i >= 0; i--)
      { temp[i] = x; x = x.r; temp[i].r = null; }
    bq = BQjoin(bq, temp);
    return v;
  }
```

links are all null). To adhere to our convention in this case, we need to add a null link at the end, which necessitates increasing the size of the array by 1. The method grow that Program 9.14 invokes for this task is simple to implement (see Exercise 9.62).

Other binomial-queue operations are also best understood by analogy with binary arithmetic. As we shall see, implementing *join* corresponds to implementing addition for binary numbers.

For the moment, assume that we have an (efficient) method for *join* that is organized to merge the priority-queue reference in its second operand with the priority-queue reference in its first operand (leaving the result in the first operand). Using this method, we could implement the *insert* operation with a call to the *join* method where one of the operands is a binomial queue of size 1 (see Exercise 9.66).

We can also implement the *remove the maximum* operation with one call to *join*. To find the maximum item in a binomial queue, we

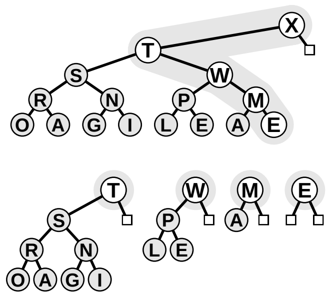


Figure 9.18
Removal of the maximum in a power-of-2 heap

Taking away the root gives a forest of power-of-2 heaps, all left-heap ordered, with roots from the right spine of the tree. This operation leads to a way to remove the maximum element from a binomial queue: Take away the root of the power-of-2 heap that contains the largest element, then use the join operation to merge the resulting binomial queue with remaining power-of-2 heaps in the original binomial queue.

scan the queue's power-of-2 heaps. Each of these heaps is left-heap-ordered, so it has its maximum element at the root. The largest of the items in the roots is the largest element in the binomial queue. Because there are no more than $\lg N$ heaps in the binomial queue, the total time to find the maximum element is less than $\lg N$.

To perform the *remove the maximum* operation, we note that removing the root of a left-ordered 2^k-heap leaves k left-ordered power-of-2 heaps—a 2^{k-1}-heap, a 2^{k-2}-heap, and so forth—which we can easily restructure into a binomial queue of size $2^k - 1$, as illustrated in Figure 9.18. Then, we can use the *join* operation to combine this binomial queue with the rest of the original queue in order to complete the *remove the maximum* operation. This implementation is given in Program 9.15.

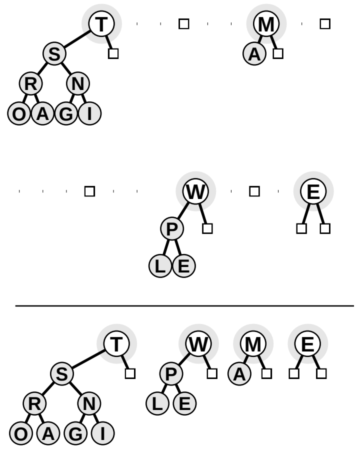


Figure 9.19
Joining of two binomial queues (no carry)

When two binomial queues to be joined do not have any power-of-2 heaps of the same size, the join *operation is a simple merge. Doing this operation is analogous to adding two binary numbers without ever encountering* $1 + 1$ *(no carry). Here, a binomial queue of 10 nodes is merged with one of 5 nodes to make one of 15 nodes, corresponding to* $1010_2 + 0101_2 = 1111_2$.

How do we join two binomial queues? First, we note that the operation is trivial if they do not contain two power-of-2 heaps of the same size, as illustrated in Figure 9.19: we simply merge the heaps from the two binomial queues to make one binomial queue. A queue of size 10 (consisting of an 8-heap and a 2-heap) and a queue of size 5 (consisting of a 4-heap and a 1-heap) simply merge together to make a queue of size 15 (consisting of an 8-heap, a 4-heap, a 2-heap, and a 1-heap). The more general case follows by direct analogy with performing addition on two binary numbers, complete with carry, as illustrated in Figure 9.20.

For example, when we add a queue of size 7 (consisting of a 4-heap, a 2-heap, and a 1-heap) to a queue of size 3 (consisting of a 2-heap and a 1-heap), we get a queue of size 10 (consisting of an 8-heap and a 2-heap); to do the addition, we need to merge the 1-heaps and carry a 2-heap, then merge the 2-heaps and carry a 4-heap, then merge the 4-heaps to get an 8-heap result, in a manner precisely analogous to the binary addition $011_2 + 111_2 = 1010_2$. The example of Figure 9.19 is simpler than Figure 9.20 because it is analogous to $1010_2 + 0101_2 = 1111_2$, with no carry.

This direct analogy with binary arithmetic carries through to give us a natural implementation for the *join* operation (see Program 9.16). For each bit, there are eight cases to consider, based on all the possible different values for the 3 bits involved (carry and 2 bits in the operands). The code is more complicated than that for plain addition, because we are dealing with distinguishable heaps, rather than with indistinguishable bits, but each case is straightforward. For example, if all 3 bits are 1, we need to leave a heap in the result binomial queue and to join the other two heaps for the carry into the next position. Indeed, this operation brings us full cycle on abstract data types: we (barely) resist the temptation to cast Program 9.16 as a purely abstract binary addition procedure, with the binomial-queue implementation nothing more than a client program using the more complicated bit addition procedure in Program 9.13.

We do a *join* operation after reducing the size of the binomial queue by 1. If the next-to-last link in the result is null, then we can shrink the size of the array by 1, since the last link in the result will be null. The method `shrink` that Program 9.16 invokes for this task is simple to implement (see Exercise 9.62).

Property 9.7 *All the operations for the priority-queue ADT can be implemented with binomial queues such that $O(\lg N)$ steps are required for any operations performed on an N-item queue.*

These performance bounds are the goal of the design of the data structure. They are direct consequences of the fact that the implementations all have only one or two loops that iterate through the roots of the trees in the binomial queue. ■

Another option, which leads to a slightly simpler implementation, is to keep the number of trees in the queue constant (see Exercise 9.64) so that the running time of all the methods is proportional to this number (the logarithm of the maximum size of the binomial queue).

Property 9.8 *Construction of a binomial queue with N insert operations on an initially empty queue requires $O(N)$ comparisons in the worst case.*

For one-half the insertions (when the queue size is even and there is no 1-heap), no comparisons are required; for one-half the remaining insertions (when there is no 2-heap), only 1 comparison is required; when there is no 4-heap, only 2 comparisons are required; and so forth. Thus, the total number of comparisons is less than $0 \cdot N/2 + 1 \cdot N/4 + 2 \cdot N/8 + \ldots < N$. ■

Binomial queues provide guaranteed fast performance, but data structures have been designed with even better theoretical performance

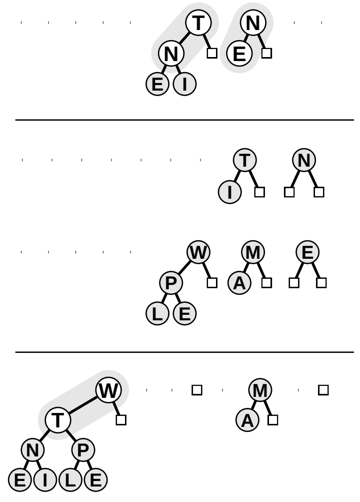


Figure 9.20
Joining of two binomial queues

Adding a binomial queue of 3 nodes to one of 7 nodes gives one of 10 nodes through a process that mimics the binary addition $011_2 + 111_2 = 1010_2$. *Adding* N *to* E *gives an empty 1-heap in the result with a carry 2-heap containing* N and E. *Then, adding the three 2-heaps leaves a 2-heap in the result with a carry 4-heap containing* T N E I. *This 4-heap is added to the other 4-heap, producing the binomial queue at the bottom. Few nodes are touched in the process.*

Program 9.16 Joining (merging) of two binomial queues

This code mimics the operation of adding two binary numbers. Proceeding from right to left with an initial carry bit of 0, we treat the eight possible cases (all possible values of the operands and carry bits) in a straightforward manner. For example, case 3 corresponds to the operand bits being both 1 and the carry 0. Then, the result is 0, but the carry is 1 (the result of adding the operand bits).

We exchange the references if necessary to make `a` reference the queue with the larger array representation, and we increase or decrease the size of `a`'s array for the result, if warranted. If there is a carry into the last (null) link in case 4, it invokes `grow`; if the next-to-last link is null after the operation is complete, it invokes `shrink`.

Like `pair`, this method is a private method in the implementation, which is called by `getmax` and `join`. The implementation of the ADT method `join(PQfull p)` is the invocation `BQjoin(bq, p.bq)`.

```
static int bit(Node x)
  { return x == null ? 0 : 1; }
static int bits(Node C, Node B, Node A)
  { return 4*bit(C) + 2*bit(B) + 1*bit(A); }
static Node[] BQjoin(Node[] a, Node[] b)
  { Node c = null;
    if (a.length < b.length)
      { Node[] t = a; a = b; b = t; }
    for (int i = 0; i < b.length; i++)
      switch(bits(c, b[i], a[i]))
        {
          case 2: a[i] = b[i]; break;
          case 3: c = pair(a[i], b[i]);
                  a[i] = null; break;
          case 4: if (i == a.length-1) a = grow(a);
                  a[i] = c; c = null; break;
          case 5: c = pair(c, a[i]);
                  a[i] = null; break;
          case 6:
          case 7: c = pair(c, b[i]); break;
        }
    if (a[a.length-1] == null) a = shrink(a);
    return a;
  }
```

characteristics, providing guaranteed constant-time performance for certain operations. This problem is an interesting and active area of data-structure design. On the other hand, the practical utility of many of these esoteric structures is dubious. Before delving into complex data-structure solutions, we need to be certain that performance bottlenecks exist that we can relieve only by reducing the running time of some priority-queue operation. Indeed, for practical applications, we should prefer a trivial structure for debugging and for small queues; then, we should use heaps to speed up the operations unless fast *join* operations are required; finally, we should use binomial queues to guarantee logarithmic performance for all operations. All things considered, however, a priority-queue package based on binomial queues is a valuable addition to a software library.

Exercises

▷ **9.54** Draw a binomial queue of size 29, using the binomial-tree representation.

• **9.55** Write a program to draw the binomial-tree representation of a binomial queue, given the size N (just nodes connected by edges, no keys).

9.56 Give the binomial queue that results when the keys E A S Y Q U E S T I O N are inserted into an initially empty binomial queue.

9.57 Give the binomial queue that results when the keys E A S Y are inserted into an initially empty binomial queue, and give the binomial queue that results when the keys Q U E S T I O N are inserted into an initially empty binomial queue. Then give the result of *remove the maximum* for each queue. Finally, give the result when the *join* operation is performed on the resulting queues.

9.58 Using the conventions of Exercise 9.1, give the sequence of binomial queues produced when the operations

P R I O * R * * I * T * Y * * * Q U E * * * U * E

are performed on an initially empty binomial queue.

9.59 Using the conventions of Exercise 9.2, give the sequence of binomial queues produced when the operations

(((P R I O *) + (R * I T * Y *)) * * *) + (Q U E * * * U * E)

are performed on an initially empty binomial queue.

9.60 Prove that a binomial tree with 2^n nodes has $\binom{n}{i}$ nodes at level i for $0 \leq i \leq n$. (This fact is the origin of the name *binomial tree*.)

▷ **9.61** Give an implementation for `empty()` that is appropriate for the binomial-queue implementation given in the text.

9.62 Implement the `grow` and `shrink` methods that increase by one and decrease by one, respectively, the size of a `Node` array leaving a `null` link in the last array position (`shrink` should throw an exception if that is not the case).

9.63 Change the binomial-queue implementation in the text to represent the queues with Java `Vector` objects instead of arrays.

∘ **9.64** Develop a binomial-queue implementation that uses a fixed-size array of representation so that `grow` and `shrink` are not needed, but the operations all take time proportional to the array size.

∘ **9.65** Modify your solution such that Property 9.7 holds, by maintaining a sentinel pointer to mark the point where the loops should terminate.

• **9.66** Implement *insert* for binomial queues by just using the *join* operation explicitly.

•• **9.67** Implement *change priority* and *remove* for binomial queues. *Note*: You will need to add a third link, which points up the tree, to the nodes.

• **9.68** Add an implementation of `clone` to the binomial-queue implementations (Programs 9.13 through 9.16) in the text, and test your implementation with your driver program from Exercise 9.44.

• **9.69** Empirically compare binomial queues against heaps as the basis for sorting, as in Program 9.6, for randomly ordered keys with $N = 1000$, 10^4, 10^5, and 10^6.

• **9.70** Develop an in-place sorting method like heapsort, but based on binomial queues. *Hint*: See Exercise 9.38.