Chapter 5

Multiplication

Multiplying by 9s has lots of tricks but I get them mixed up.

We're learning times . . . I can't remember the answers all the time but it's ok because I got a calculator.

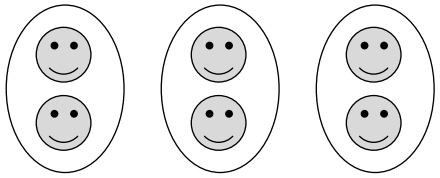
We're supposed to move the numbers over when we multiply by lots of numbers but I don't know why . . . I just do it because the teacher said to and it works.

If you multiply 3 apples by 4 apples, you get 12 apples.

Multiplying is hard for me because I have to make all the little marks and sometimes there are so many I get mixed up.

What Is Multiplication?

Multiplication of whole numbers is represented in the physical world by unioning multiple sets of equal cardinality. An example is a drawing of three sets of 2 objects:

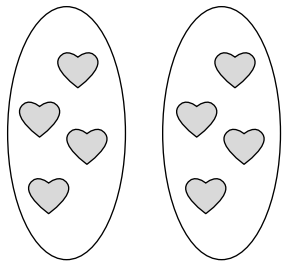


3×2

The representation of the arithmetical operation for expressing this unioning process is multiple addition. Thus, two plus two plus two is written as three 2s, using the notation 3×2. Early work with multiplication should mainly be devoted to the conceptual understanding that multiplication is a shorthand notation for denoting multiple addition. Multiplication situations should be presented to children, and they should then use materials (beans, counters, etc.) to demonstrate the problem given and to generate an answer (a product).

Do not be concerned with habituating the facts too early. It is essential that children understand that the multiplication problem expresses a relationship between the numbers involved and that they own the meaning of the symbolism—that the first factor in the problem denotes the number of sets and the

second factor denotes the number of objects contained in each set. The product is then the total number of objects when the sets are joined (unioned). For example, consider the following problem: Two children each have four hearts on Valentine's day. How many hearts are there altogether? This problem is then expressed using materials, by constructing two sets with 4 hearts in each set. The shorthand way of writing this is 2×4, which results in 4 plus 4, or 8 hearts altogether.



2×4

As children progress in conceptualizing the meaning of multiplication and can express what the multiplication situation means, it is time to begin habituating the facts. We do not like the word "memorizing" because it communicates that the fact is a paired association, of 2 with 4 in the above example, but the student could be devoid of an understanding of what the multiplication symbol means. Instead, after much work with material objects, the association of 2 with 4 is habituated so the child can develop speed and accuracy in recalling the fact. Habituation does not contribute to understanding; it presumes that understanding precedes the habituation, and the fact is habituated in order for the child to gain speed in working with multiplication situations.

What Should Students Understand About Multiplication?

In grades 3–5, a central focus should be directed at helping children develop the conceptual meaning for whole number multiplication and division (NCTM, 2000). Multiplication and division can begin to have meaning in earlier grades by engaging children in problem situations that utilize multiplication concepts for solution. Further, developing fluency involves a connection and a balance between conceptual understanding and computational proficiency. Understanding without fluency can inhibit children's problem solving abilities (Thornton, 1990). When discussing computational strategies by developing, recording, and discussing each other's strategies, important kinds of learning can occur (NCTM, 2000). The Standards also point out that children must become fluent in computation and must have efficient, accurate methods supported by an understanding of numbers and operations. On the other hand, overpracticed computational procedures that are devoid of conceptual understanding are an undesirable classroom practice (NCTM, 2000, p. 35).

In this chapter you will continue an error analysis approach. This is one of the steps to be used in designing a DAS and a MIP for a student. A simulated child's paper will be presented and you should follow the directions in working through each of the three examples. The prescription proposed will direct the instructional design suggested in the MIP.

About the Student: Alberto

Alberto is a bright sixth grader who likes school. He enjoys learning activities and the social aspect of school. He is open and friendly with peers but is more reticent around adults. He has a very strong sense of fairness and is well liked by both boys and girls. He has excellent coordination and prefers an active life, which makes it difficult for him to sit still for long periods and stay focused on routine tasks. He generally follows school and classroom rules and rarely receives a negative behavior mark. He does not like to make mistakes and is slow to complete assignments because he tries to get everything done perfectly. Rather than turn in incomplete work, he will put it in his desk. Alberto does not like to ask for help because he does not want to look "stupid."

His reading skills are appropriate for sixth grade; however, Alberto dislikes reading and delays tasks involving reading. When avoidance tactics are unsuccessful, he reads quickly and with little attention to nuances and detail. Consequently, he misses important information. Alberto's reading style causes him to misinterpret directions. His reading style also causes him to think he has read one thing when that is not what was written, resulting in his answers being incorrect. He then becomes frustrated and angry with himself but does not acknowledge the connection between his approach to reading and the resulting errors in his work in math.

Alberto quickly memorizes math facts. He knows addition, subtraction, multiplication, and division facts and is learning a division algorithm. He is concrete in his thinking, which slows his understanding of abstract concepts. Once he has grasped a concept, he retains the information. Alberto's rapid reading and skimming of details causes problems for him in math. He may not attend to all the steps in a multistep problem, which results in a poor grade. When solving word problems, Alberto's reading creates numerous problems for him. When confronted with his errors he becomes frustrated, which results in self-deprecating comments.

Alberto enjoys helping his peers but especially enjoys helping adults. He likes to feel useful but quickly sees through "made up" jobs. He likes to earn free time, which he usually spends drawing or playing games with friends. Alberto does not respond to tangible reinforcers, such as stickers or points, unless the points lead to free time or helping time.

Alberto prefers learning tasks that require active participation or a hands-on approach. He enjoys class discussion and is an active participant in group work. He gets upset if he perceives a group member is not doing their share of the work or is "goofing off." Alberto is able to put his ideas in written form, although he has difficulty getting started. Once he gets the first sentence down, he usually has little difficulty completing the task. He especially likes giving reports and does not mind being in front of the group.

Error Patterns: Diagnosis, Prescription, Remediation

Each of the following simulations assumes that Alberto has taken a test, and in each situation, a different error has emerged. The task is to identify appropriate lessons based on the knowledge of the child and your knowledge of the exhibited math deficiency.

Multiplication Error Pattern I for Alberto

The first sample paper is a completed, nine-problem multiplication paper. You should use the following four-step process:

1. Score the paper.
2. Begin with the first incorrect problem and attempt to determine the algorithm Alberto is using to obtain the answers.
3. Use Alberto's pattern and see if the second incorrect problem follows the pattern.

STUDENT WORK SAMPLE ERROR I FOR ALBERTO

1. $\overset{3}{1}7 \times 5 = 85$

2. $40 \times 8 = 320$

3. $\overset{1}{2}3 \times 4 = 92$

4. 27×31
   27
   81
   837

5. $\overset{3}{5}4 \times 19$
   486
   84
   1326

6. $\overset{4}{5}6 \times 28$
   448
   142
   1868

7. $4\overset{1}{0}5 \times 63$
   1215
   2410
   25315

8. $\overset{2}{1}\overset{1}{4}3 \times 806$
   858
   1034
   11198

9. $\overset{1}{9}30 \times 884$
   3720
   7340
   7340
   811120

   a. If it does, go to step four.
   b. If it does not, study the error in the second problem and revise your prediction of the algorithm used.
4. Confirm the pattern using the third incorrect problem.

Complete the four-step process now, using the data shown.

Diagnosing the Error: Once you have finished scoring the paper and identifying the proposed pattern of error, you should reflect on the strengths exhibited in the child's multiplication work. Strengths to look for include knowledge of place value, facts (both easy [products involving 2, 5, 3, and 4]) and hard ([products involving 6, 7, 8 and 9]), work with zero in multiplication, and the ability to regroup in multiplication—to mention just a few. Use the space below to record your observations.

Alberto's Error Pattern(s):
Alberto's Strengths:

Scoring Work Sample Error I for Alberto finds him getting four of nine (44 percent) correct. While he has a failing score, he has a number of strengths which should prove beneficial in any remediation. He knows the column format for multiplication and he knows that he should begin multiplying by the ones digit of the multiplier. Alberto also knows:

1. easy multiplication facts involving 2s, 5s, 3s, and 4s (he gets all of these products correct)
2. multiplication involving hard facts (6s, 7s, 8s, and 9s)
3. multiplication when zero or one are involved
4. he shifts to the tens column when multiplying by the tens digit in the multiplier, and so on.

He seems to add the partial products obtained from each subpart of the multiplier. Not a single product is incorrect in the use of his multiplication facts to obtain the partial products. This child is really quite strong in multiplication and surely feels frustrated getting incorrect answers when doing so much correct.

Prescription: On each problem, this student seems to perform the first part of the multiplication correctly when multiplying by the units digit of the multiplier. He regroups correctly and writes the digit to be regrouped in the correct place throughout the first iteration.

It is when he begins to multiply by the tens digit (as in problem 5) that the error begins to occur. He has not erased or marked out the previous multiplications. So when he multiplies the 1 by the 5, he adds in the digit from the

previous multiplication, thus getting 1×5 plus 3, yielding the 8. His error seems to be that he is reusing values from previous multiplications. Look at the next incorrect problem to see if this is the case.

Problem 6 reveals that, when he begins to multiply by the 2 in the tens column of the multiplier, he obtains 2 times 6 equals 12. He records the 2 in the tens column but does not change the regrouped 4 from the previous product to the new 1 in the 2×6 result. Then he multiplies 2 times 5, which is 10, and adds on the 4 from the previous multiplication sequence. This yields the 14, which he records on the second line of the partial product.

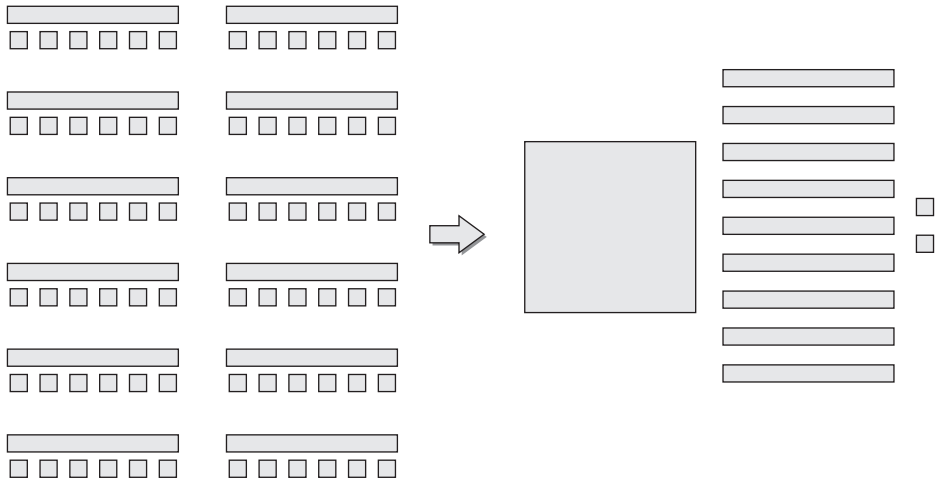
Checking problem 7 confirms the error diagnosis. When multiplying by the 6 in the tens digit, he multiplies 6 times 5 and records the 0 in the 30. Instead of regrouping the 3 from the 30, he multiplies the 6 times 0 and adds the 1 from the previous partial product. He then records the 1 in the partial product and moves on to multiply the 6 times 4 and records the 24.

A complication occurs in problem 8. When multiplying by the 0 in the tens digit of the multiplier, he fails to record the zero product, thus causing the multiplication by the 8 in the multiplier to be recorded in the wrong columns. This will require some special attention, but is independent of the primary error exhibited throughout the test. A completed DAS for Alberto is included (see Table 5.1).

Remediation: Alberto's first MIP appears in Table 5.2. Begin working with this child by working a problem like the following:

$$\begin{array}{r} 16 \\ \times\ 12 \\ \hline \end{array}$$

Have the child work the problem with pencil and paper. The child should obtain an answer of 292 if he uses the incorrect procedure exhibited on his test. Then have the child work the problem using base ten blocks or popsicle sticks.



Alberto should arrive at 192 using the base ten blocks. At this point, ask why the two answers are different. Listen carefully to the explanation,

TABLE 5.1 Data Analysis Sheet

Student: Alberto						
Team Members:						
Context	**Content Assessment**	**Process**		**Behavior**		**Reinforcement**
		Input	Output	Academic	Social	
+	+	+	+	+	+	+
• Likes school • Open and friendly with peers • Likes to be clustered with his peers • Enjoys all cooperative learning activities • Enjoys any and all group participatory activities • Likes being in front of class	***Learned Concepts I*** • Knows easy facts involving 2s, 5s, 3s, & 4s • Knows hard facts involving 6s, 7s, 8s, & 9s • Multiplies correctly when zero or one is a factor • Correctly uses place value format in multiplication ***Learned Concepts II*** • Knows easy facts involving 2s, 5s, 3s, & 4s • Knows hard facts involving 6s, 7s, 8s, & 9s • Multiplies correctly when zero or one is a factor • Works well with zero as a multiplicand ***Learned Concepts III*** • Knows easy facts involving 2s, 5s, 3s, & 4s • Knows hard facts involving 6s, 7s, 8s, & 9s • Multiplies correctly when zero or one is a factor • Correctly marks out previously regrouped digits when performing partial products	• Reading skills are grade appropriate • Memorizes quickly • Concrete thinker • Understands abstract concepts	• Excellent coordination • Prefers an active life style • Retains information learned • Enjoys group work • Is good at putting ideas into written form • Once he gets started he rarely fails to complete his work • Likes giving oral reports	• Enjoys learning activities • Likes concrete activities • Wants to complete his activities • Good at math	• Enjoys social aspects of school • Strong sense of fairness • Well liked by boys and girls • Follows school rules	• Likes being with peers • Enjoys helping adults • Likes to feel useful • Free time is important to him • Likes drawing and playing games with his friends • Points help if they lead him to free time or helping time • Likes being in front of class
–	–	–	–	–	–	–
• Reticent around adults • Can't sit for a long period of time • Doesn't like to work with his teacher on a one-on-one basis • Doesn't like to work alone • Doesn't like to be isolated in the classroom	***Error Pattern I*** • When multiplying by tens or hundreds digit, reuses previously renamed values • Fails to record zero product when zero is one of the factors ***Error Pattern II*** • Fails to record regrouped digit in partial products ***Error Pattern III*** • When regrouped digits are involved in a partial product, adds the regrouped digit to the factor before multiplying by the multiplier • Adds before multiplying rather than multiplying then adding the regrouped digit	• Does not like to ask for help • Does not like reading • Misinterprets directions • Thinks he has read something when he hasn't • Doesn't do well with multitask assignments	• Slow to complete assignments—rather than turn in incomplete assignments he will put his work in his desk uncompleted • He reads quickly with little attention to material • Misses important information • Has trouble with word problems (reading related)	• Slow to complete assignments—rather than turn in incomplete assignments he will put his work in his desk uncompleted • He reads quickly with little attention to material • Misses important information • Has trouble with word problems (reading related)	• Rarely breaks school rules • Becomes frustrated and angry when he makes errors • Becomes upset when he believes a group member is not doing his share	• Does not like made-up work, nor does he like tangible reinforcers like stickers and candy • Avoids independent reading • Doesn't like being with teacher one-on-one

Note: The + symbols indicate strengths and the – symbols indicate areas of concern.

TABLE 5.2 Mathematics Improvement Plan I for Alberto: Correctly Multiplying by the Tens Digit

Time		15 minutes	20–30 minutes	20 minutes
Context		Classroom activities of a cooperative type, he is in a group activity (+)	Independent seat work (–)	He will work on this activity in the classroom with a classmate whom he likes (+)
Content		Group works multiplication problems (1-digit multiplier) using manipulatives (+)	Works and records exercises like those on test, uses base-10 blocks, to check results, practices multidigit products requiring regrouping (–)	Group of 4 children plays Rollette activity (+)
Process	Input	Teacher gives multiple instructions for the task at hand (–)	The work has visual and written examples of the task at hand (+)	The written directions are at grade level and have a singular task (+)
	Output	Group is engaged in manipulative activities that requires a concrete outcome (+)	Student is expected to write his results and read them to the entire class (–)	The words in the word problem are at a challenging level (–)
Behavior	Academic	Group produces a single written project with all having input (+)	Student completes his work even though it has many errors (–)	He consistently has to ask questions of the teacher to complete the task (–)
	Social	Group works well together and all students have a sense of their responsibility (+)	Student becomes easily frustrated and does poorly at the beginning of his work (–)	His classmate is one whom he likes and he trusts (+)
Reinforcement		If the group does well they will get free time to play games with their partners (+)	For every problem he gets right he will get to help the teacher at a needed task (+)	Teacher gives all the students a smile face as a reward for their work (–)

Note: The + symbols indicate strengths and the − symbols indicate areas of concern.

because it will give clues to how you will discuss correcting the error. Probably, Alberto will think he has done something wrong but will not be sure which of the procedures has the error. We would suggest that you rework the problem with Alberto, with you recording information as the child performs the multiplication with the base ten blocks:

$$\begin{array}{r} 16 \\ \times\ 12 \\ \hline \end{array} \quad \textit{means} \quad \begin{array}{r} 16 \\ \times\ 10 \\ \hline \end{array} \quad + \quad \begin{array}{r} 16 \\ \times\ \ 2 \\ \hline \end{array}$$

Using the base ten blocks, point out to Alberto that 16 times 12 is 16 times 10 plus 16 times 2. Begin with the 2 and move 2 groups of 16 apart from the other 10 groups. Ask him to determine how many 2 times 16 are, with the base ten blocks. The child should give 32 as the answer. Now you record the value:

$$\begin{array}{r} 16 \\ \times\ 12 \\ \hline 32 \end{array} \quad \textit{means} \quad \begin{array}{r} 16 \\ \times\ 10 \\ \hline \end{array} \quad + \quad \begin{array}{r} 16 \\ \times\ \ 2 \\ \hline \end{array} \qquad \leftarrow \textit{(you write this)}$$

Now have Alberto determine what 10 times 16 is, using the base ten blocks. He should answer 160. Record the value:

$$\begin{array}{r} 16 \\ \times\ 12 \\ \hline 32 \\ 160 \\ \hline \end{array} \quad \textit{means} \quad \begin{array}{r} 16 \\ \times\ 10 \\ \hline \end{array} \quad + \quad \begin{array}{r} 16 \\ \times\ 2 \\ \hline \end{array}$$

← *(you write this)*

Have Alberto sum the two partial products to find the total. He will probably arrive at 192. Then it should be apparent that the answer obtained using the base ten blocks is the correct one. At this point, Alberto should be ready to compare the way he worked the problem with pencil and paper with the answer just obtained. Encourage him to explain how he worked the problem, comparing the result to what was just obtained:

$$\begin{array}{r} 16 \\ \times\ 12 \\ \hline 32 \\ 160 \\ \hline 192 \end{array} \qquad \begin{array}{r} {}^{1}16 \\ \times\ 12 \\ \hline 32 \\ 26 \\ \hline 292 \end{array}$$

Alberto should arrive at the same first partial product of 32. However, on the second partial product, his invented algorithm will lead to 26 instead of 16. At this point, ask why he is using the regrouped 1 in this product. Alberto is very bright, and will realize immediately that the 1 has no place in this product—and will also realize that this mistake is leading to his error. Discuss with him what he thinks is the best way to not reuse the digit from the first product. In our experience, he will either say "mark it out" or "erase it." We would encourage you to suggest that it is best to mark it out. The reason is that, when children write a problem, then try to erase part of the work, they invariably smudge the problem and have trouble in further work with the problem. Marking it out will accomplish the same thing and will not lead to complications in further work with the problem.

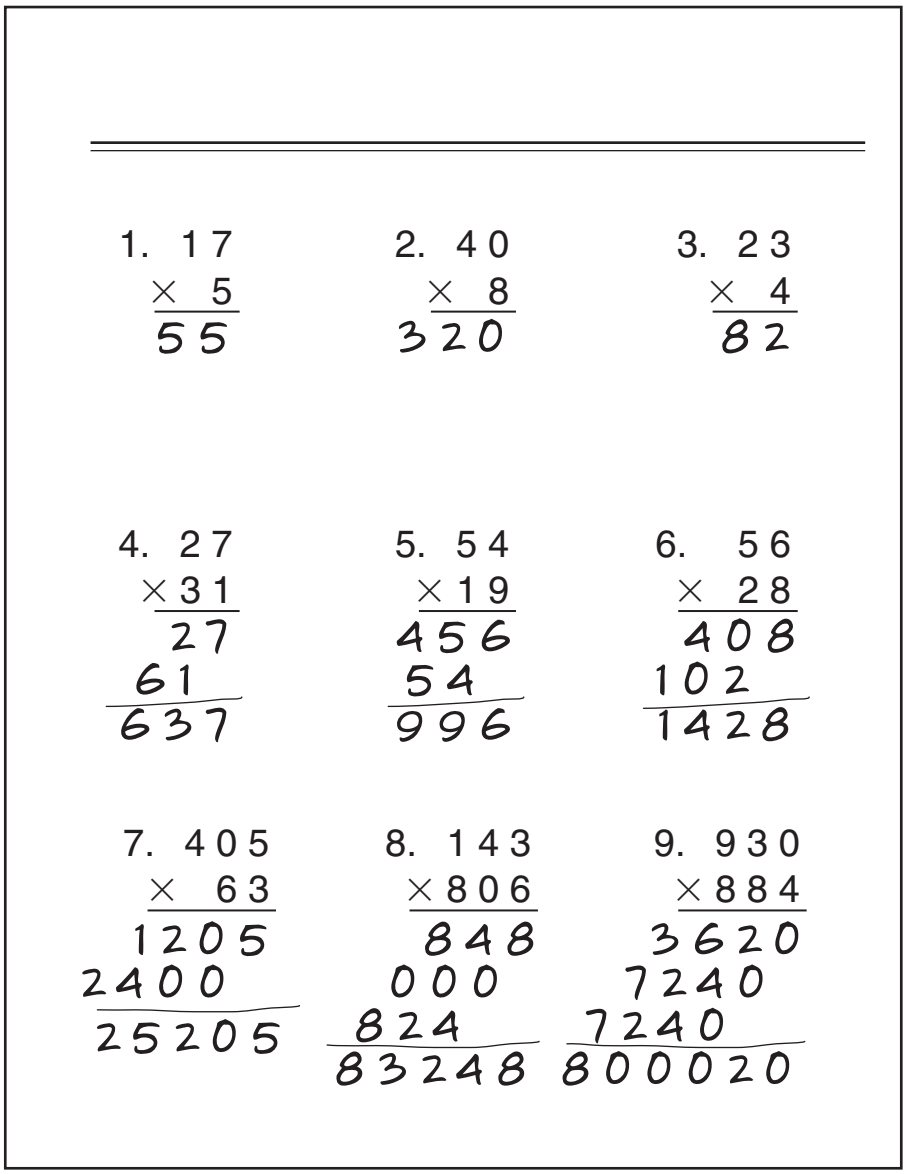
This child has so many strengths that he will be quite relieved to finally determine why he is getting so many problems wrong when he thinks he is doing the problem correctly. You should work another problem that he got incorrect on the test and let him determine where he made the error. When you are working with a child, give extra credit for finding the errors if it makes him more interested in his math work.

Multiplication Error Pattern II for Alberto

Alberto's second paper is also a nine-problem multiplication paper. You should complete the four-step process outlined for Error Pattern II (see page 92).

Diagnosing the Error: Once you have finished scoring the paper and identifying the proposed pattern of error, you should reflect on the strengths exhibited in Alberto's multiplication work. Strengths to look for include knowledge of place value, facts (both easy [products involving 2, 5, 3, and 4] and hard

STUDENT WORK SAMPLE ERROR II FOR ALBERTO



[products involving 6, 7, 8, and 9]), work with zero in multiplication, and the ability to regroup in multiplication. Use the space below to record your observances.

Alberto's Error Pattern(s):
Alberto's Strengths:

Alberto scored one correct of the nine problems (11 percent). While this performance seems quite low, he exhibits tremendous strength in multiplication and should improve his performance immediately. Alberto knows:

1. easy multiplication facts involving 2s, 5s, 3s, and 4s (he gets all of these products correct)
2. multiplication involving hard facts (6s, 7s, 8s, and 9s)
3. multiplication when zero or one are involved. He especially does well with zero in problem 8
4. he shifts to the tens column when multiplying by the tens digit in the multiplier, and so on
5. he works well with zero when it appears in the multiplicand, as in problems 2, 7, and 9

The first incorrect problem, problem 1, is not regrouping when the product results in a value larger than 10. Alberto seems to drop the regrouped digit and just writes the units value of the product. Looking at the next error, problem 3, reveals the same error—but maybe it is because the digit to be regrouped is a 1. Investigating the third incorrect problem, problem 4, discloses that, in the second partial product, he does not record the 2 that is regrouped from the 3×7 product. Thus, he is just not recording the regrouped digit—regardless of its value.

Prescription: Alberto needs to record the regrouped digit in all partial products.

Remediation: Alberto's second MIP appears in Table 5.3. Begin by asking Alberto to work problem 1 using base ten materials. He should get 5 groups of 1 long (10) and 7 units. Now have him determine the total. He should get 85. Have him compare that with his answer on the test. This should put him in a state of disequilibrium according to Piaget, and he is at a point where he is ready to determine where the discrepancy is.

There are a number of ways to proceed at this point; you could have Alberto complete the problem while you record what is obtained. Have him gather his 5 groups of 17 and begin the problem solution again. You write the problem out so you can record as he works the problem.

Ask him to accumulate the 5 groups of 7 units. He will determine that it is 35, and will be able to show you. Encourage him to trade the units in for longs so he has 3 longs and 5 units at the end. You record this as:

$$\begin{array}{r} 17 \\ \times\ 5 \\ \hline 35 \end{array} \quad \leftarrow \quad \textit{(you record this)}$$

Now have him accumulate the longs (10s) and tell you that the value is 50. You record this as:

$$\begin{array}{r} 17 \\ \times\ 5 \\ \hline 35 \\ 50 \\ \hline 85 \end{array} \quad \leftarrow \quad \textit{(you record this)}$$

At this point, ask Alberto what happened to the 3 longs. He should indicate that he left them out. Point out to him why we write the regrouped value up to

TABLE 5.3 Mathematics Improvement Plan II for Alberto: Recording the Regrouped Digit in all Partial Products

Time		15 minutes	20 minutes	20 minutes
Context		Classroom activities of a cooperative type, he is in a group activity (+)	Independent seat work (–)	He will work on this activity in the classroom with a classmate whom he likes (+)
Content		Group generates multiplication problems and records process (problems should require regrouping on some problems) (+)	He works problems like on the test. He generates the problem with base ten blocks, then draws a picture of the problem. He constructs his "my multiplication" book (–)	He and classmate will play "200 to Win" (+)
Process	Input	Teacher gives multiple instructions for the task at hand (–)	The work has visual and written examples of the task at hand (+)	The written directions are at grade level and have a singular task (+)
	Output	Group is engaged in manipulative activities that requires a concrete outcome (+)	Is expected to write his results and read them to the entire class (–)	The problems in the activity are at a challenging level (–)
Behavior	Academic	Group produces a single written project, with all students having input (+)	Completes his work even though it has many errors (–)	He consistently has to ask questions of the teacher to complete the task (–)
	Social	Group works well together and all students have a sense of their responsibility (+)	Becomes easily frustrated and does poorly at the beginning of his work (–)	His classmate is one whom he likes and he trusts (+)
Reinforcement		If the group does well, they will get free time to play games with their partners (+)	For every problem he gets right he will get to help the teacher at a needed task (+)	Teacher gives all the students a smile face as a reward for their work (–)

Note: The + symbols indicate strengths and the − symbols indicate areas of concern.

the left of the tens digit in the multiplicand. Now have him rework the problem while you write what the values are. In the first product, 35, record the 5 and write the regrouped digit above and to the left of the 1 in the 17, as follows:

$$\begin{array}{r} {}^{3}1\,7 \\ \times\ \ 5 \\ \hline 8\,5 \end{array}$$

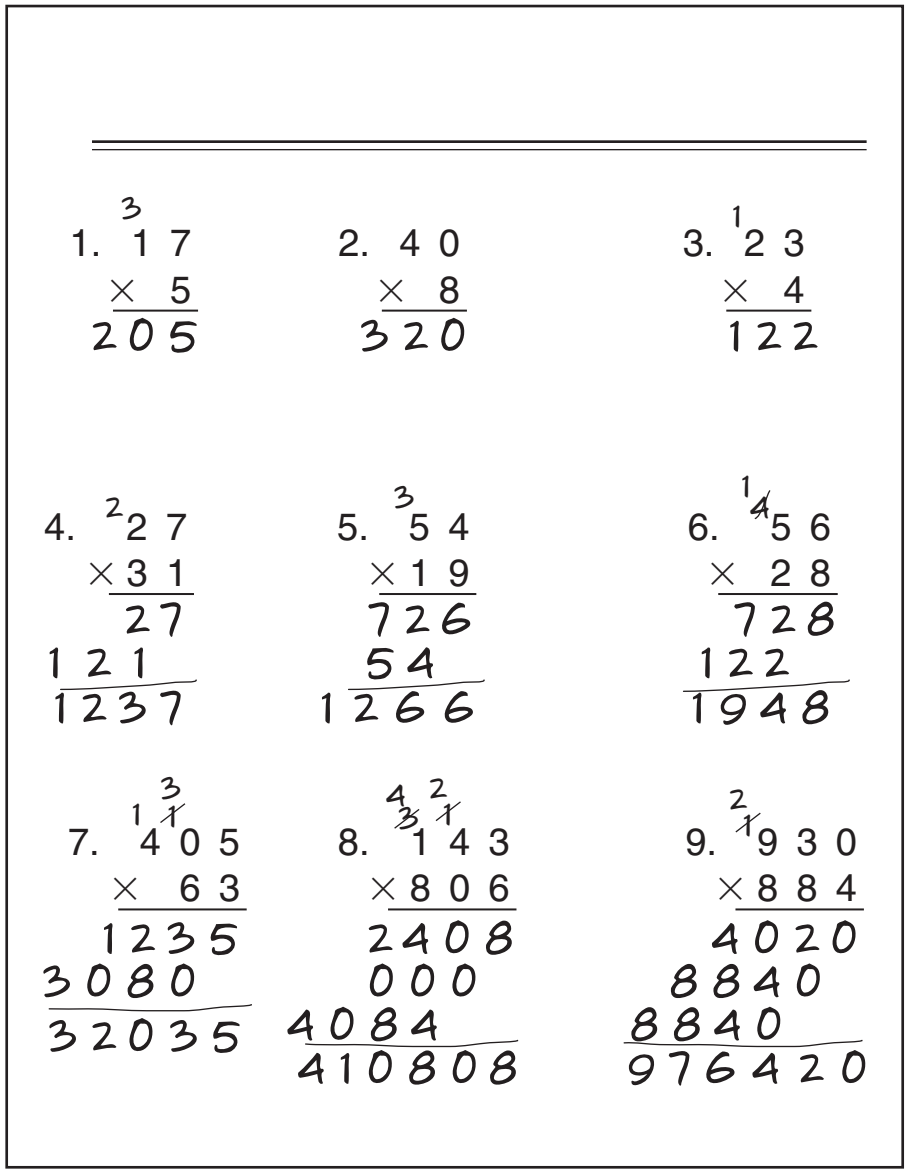
Have Alberto rework problem 3. Encourage him to talk it out as he does it. Be sure to reinforce writing the regrouped digit when the product of 3×4 is stated. He should write the 2 as before, and then write the regrouped 1 above and to the left of the 2 in 23.

The tremendous strength exhibited by Alberto in multiplication knowledge should make this a rather easy error to correct. Point out to him how much he has done correctly on the test, and how writing the regrouped digits will result in his getting all the problems correct in the future. Have him rework the problems on the test and discover his errors. Give him free choice time for correcting all the errors, as his DAS indicates this is a meaningful reward for him.

Multiplication Error Pattern III for Alberto

The third paper is similar to the first two. You should complete the same four-step process used for Error Patterns I and II (see page 92).

STUDENT WORK SAMPLE ERROR III FOR ALBERTO



Diagnosing the Error: Once you have finished scoring the paper and identifying the proposed pattern of error, you should reflect on the strengths exhibited in the child's multiplication work. Strengths to look for include knowledge of place value, facts (both easy [products involving 2, 5, 3, and 4] and hard [products

involving 6, 7, 8, and 9]), work with zero in multiplication, and the ability to regroup in multiplication. Use the space below to record your observations.

Alberto's Error Pattern(s):
Alberto's Strengths:

Alberto scores just one of nine (11 percent) correct, which is a failing performance. Yet, his work reveals numerous strengths in multiplication. First, he knows the format in multiplication. He also knows:

1. easy multiplication facts involving 2s, 5s, 3s, and 4s (all of these products are correct)
2. multiplication involving hard facts (6s, 7s, 8s, and 9s)
3. multiplication when zero or one are involved (he does well with zero, as evidenced in problem 8)
4. when multiplying by the tens digit in the multiplier, he records in the tens column, and so on
5. When needing to regroup a partial product, he crosses out the previous digit and records the new value to be used

The first incorrect problem, problem 1, reveals that Alberto adds the 3 and the 1 to get 4, then multiplies that result by 5, resulting in a value of 20. Looking at the next incorrect problem, problem 3, he again appears to add the 1 and the 2 to get 3, then multiplies by 4 to obtain the 12 recorded in his answer. Looking at problem 4 confirms that, during the second partial product, 3 is multiplied by 7 to obtain 21, the 1 is recorded, the 2 tens are regrouped, and the 2 is written just above the 2 in the multiplicand. Then 2 is added to 2 to obtain 4, which is multiplied by the 3 to obtain 12. The 12 is recorded in the second partial product.

Prescription: Alberto's error appears to be a procedural error in which he regroups part of a product, then adds the regrouped digit before multiplying, rather than multiplying the values before adding the regrouped number. It appears that he is applying a process and not thinking about the conceptual implications of the process applied. Because he exhibits so many strengths (such as place value in partial products, work with zero in the multiplier and the multiplicand, and working with regrouped values in problems 7 and 8), we believe correcting this error will not prove difficult.

This is a case in which a discussion with the child will prove invaluable in assessing the extent to which the child is cognitively deficient or procedurally incorrect. We will assume that the difficulty is procedural for the discussion.

Remediation: Alberto's third MIP appears in Table 5.4. The remedial process would begin with work on a problem like problem 1. Have Alberto get five groups of 17, using the base ten blocks. Ask him to determine how many there are altogether. He should be able to arrive at 85. Ask him to compare that answer to the answer on his test. This should provide you with the opportunity to suggest

TABLE 5.4 Mathematics Improvement Plan III for Alberto: Correcting Procedural Errors when Regrouping

Time		15 minutes	20 minutes	30 minutes
Context		Classroom activities of a cooperative type, he is in a group activity (+)	Independent seat work (–)	He will work on this activity in the classroom with a classmate whom he likes (+)
Content		Group plays "Next Move," where each child provides the next step in working a problem—they say the step then the group does it (+)	Works problems similar to those on the test, practices order of procedure in multiplication with regrouping (–)	Group plays either "Zero Wins" or "200 Wins"
Process	Input	Teacher gives multiple instructions for the task at hand (–)	The work has visual and written examples of the task at hand (+)	The written directions are at grade level and have a singular task (+)
	Output	Group is engaged in manipulative activities that requires a concrete outcome (+)	Student is expected to write his results and read them to the entire class at some point (–)	The words in the problem are at a challenging level (–)
Behavior	Academic	Group produces a single written project with all students having input (+)	Student completes his work even though it has many errors (–)	He consistently has to ask questions of the teacher to complete the task (–)
	Social	Group works well together and all students have a sense of their responsibility (+)	Student becomes easily frustrated and does poorly at the beginning of his work (–)	His classmate is one whom he likes and he trusts (+)
Reinforcement		If the group does well, they will get free time to play games with their partners (+)	For every problem he gets right he will get to help the teacher at a needed task (+)	Teacher gives all the students a smile face as a reward for their work (–)

Note: The + symbols indicate strengths and the − symbols indicate areas of concern.

reworking the problem with the base ten blocks and comparing the situation to what was done with pencil and paper.

Once again, get the five groups of 1 long and 7 units. In the meantime, you might write the problem on a side sheet so that you and he can record his work. We would use something like:

	f	l	u	*
		1	7	
×			5	

*f = flats
l = longs
u = units

Ask Alberto what the first thing he did on the pencil and paper work was. He should tell you that he multiplied the 5 by the 7. Suggest that he multiply 5 times 7 by combining the 5 groups of 7 units. Let him exchange until he gets the 35 as 3 longs and 5 units. Now ask what he wrote on the pencil and paper work. He should indicate that he wrote the 5 to represent the 5 units and

regrouped the 3 tens and wrote that above the 1 in the longs column. You should record that now.

	f	l	u
		$^{3}1$	7
×	__	__	5
			5

Have him set the 3 longs off to the side to represent regrouping the 30 units as 3 longs. Continue by asking him to find out how many longs are in the 5 groups. He should answer that there are 5 longs, plus the 3 from the regrouping, for a total of 8 longs.

Then ask him to compare what was just done with the work on his pencil and paper test. Can he tell you how the two are different? He will probably recognize that he added the 3 to the 1 before multiplying by the 5. Bright students (such as this student) will recognize, almost immediately, they have used an incorrect procedure. Quite often you will see this sort of need to refresh the algorithmic procedure if it has been a long period of time since they last used the procedure. Have him rework a problem like problem 3 and see if he can discover what he did incorrectly. It is very reinforcing for students who cannot determine why they are getting incorrect answers (when they feel they know multiplication so well) to determine exactly where they have gone astray when working problems. Have Alberto recheck his work on the other problems on the pencil and paper test and reward him with free choice time for finding the errors and correcting them.

We believe that children do not get wrong answers to frustrate you. It happens because they just do not know how to get correct answers. It will be so gratifying to these students to realize what they are doing incorrectly. With sustained practice, the correct procedure will become second nature to them.

Conclusions: Instructional Strategies Summary

Throughout the remediation plan for all three samples, the following suggestions would be very helpful:

1. Have learners complete a few problems at a time. Once you are satisfied that they understand something, move on.
2. Once an error has been corrected by the learner, have her practice ample examples to extinguish the incorrect procedure.
3. You might pair a slow learner with an average learner. They can help each other and both will profit from working together. Combine older and younger peers if appropriate.
4. Use the simplest possible numbers to explain a mathematical operation, then move to the more complex level. Always begin with simple examples before moving to complex examples.

Instructional Activities

Assessing whether children's multiplication errors are primarily results of a low level of conceptual understanding or a lack of ability to recall basic facts or work with algorithmic rules can be accomplished with various types of questions. As the teacher, you will develop a sense of what to ask and how to ask it in order to gain insight into the child's thinking. Children's conceptual learning can be assessed and evaluated more readily utilizing authentic, real world situations in which learners must decide not only how to solve the problem but whether the result is reasonable.

The following are examples of activities that can help in work with multiplication. Each is described with any special consideration you, as the teacher, need to know in order to optimize use with children.

ACTIVITY: Rollette (0 to 30)

Students should know how to generate facts by using materials and should be ready to practice habituating facts.

Objective: Habituate multiplication facts involving 0, 2, and 5

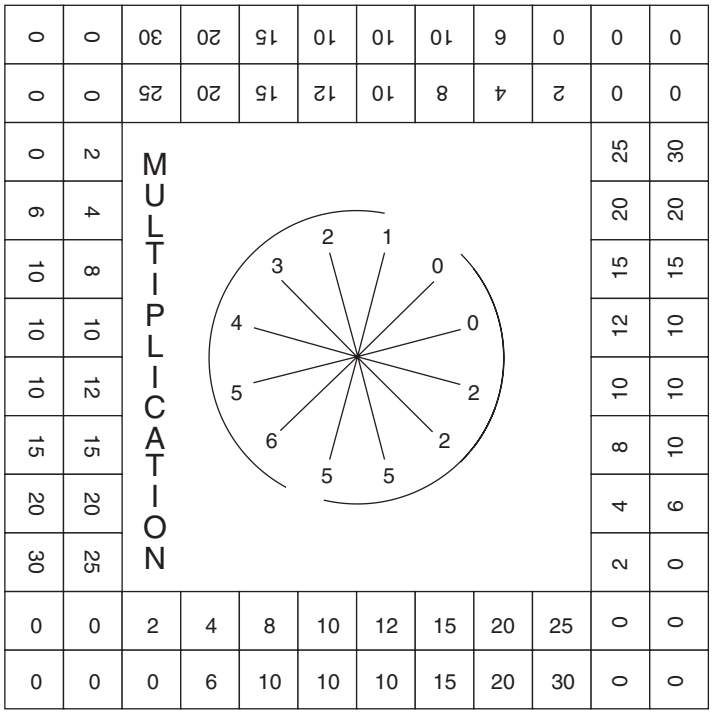
Materials:

Pencil

Sheet displayed in Figure 5.1

Two number cubes (one cube with the numbers 0, 2, and 5 and one cube with numbers 1–6)

FIGURE 5.1



Directions: Four children play, and each child is responsible for his side of the board. The player throws the two cubes and says the fact. That player then marks out that number on his side of the board. Play then passes to the player on the left. The first person to mark out all facts on his side of the board is the winner.

ACTIVITY: Rollette (0 to 30)

Students should know how to generate facts by using materials and should be ready to practice habituating facts.

Objective: Habituate facts involving products to 30

Materials:

Pencil

Sheet as displayed in Figure 5.2

Two number cubes (one cube with numbers 0–5; one cube with numbers 1–6)

Directions: Four children play, and each child is responsible for her side of the board. The player throws the two cubes and says the fact. That player then marks out that number on her side of the board. Play then passes to the player on the left. The first person to mark out all facts on her side of the board is the winner.

ACTIVITY: Rollette (3 to 48)

Students should have generated facts using materials and should be ready to practice habituating facts.

FIGURE 5.2

1	0	30	24	18	15	12	9	6	4	2	1
2	0	25	24	16	12	10	8	5	3	0	0
4	3									25	30
6	5									18	24
9	8									16	24
12	10									12	15
15	12									10	12
18	16									8	9
24	24									5	6
30	25									3	4
0	0	3	5	8	10	12	16	24	25	0	2
1	2	4	6	9	12	15	18	24	30	0	1

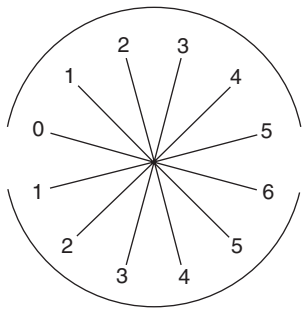
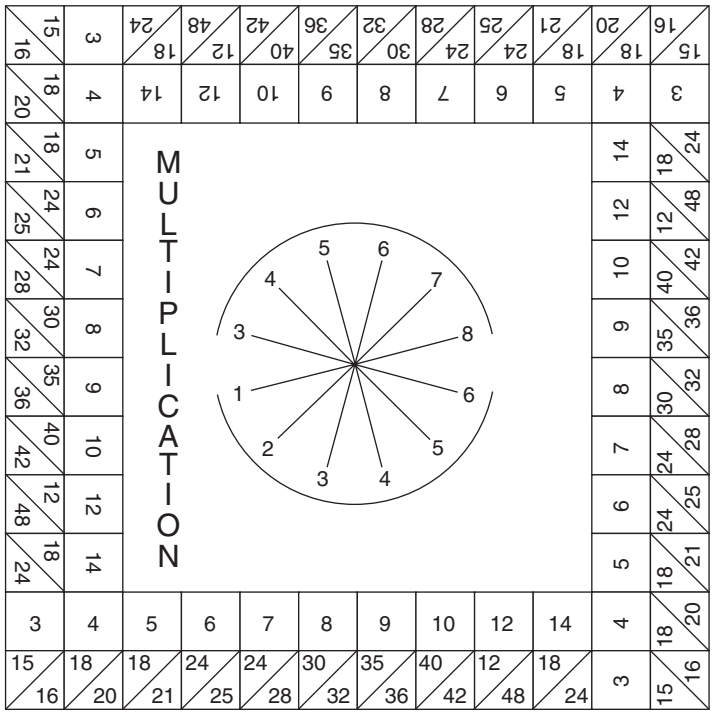


FIGURE 5.3



Objective: Habituate facts

Materials:

Pencil

Sheet as displayed in Figure 5.3

Two number cubes (one cube with numbers 3–8; one cube with numbers 1–6)

Directions: Four children play, and each child is responsible for his side of the board. The player throws the two cubes and says the fact. That player then marks out that number on his side of the board. Play then passes to the player on the left. The first person to mark out all facts on his side of the board is the winner.

ACTIVITY: 200 Wins

Students should have generated facts using materials and should be ready to practice facts.

Objective: Practice multiplication facts and maintain column addition skills

Materials

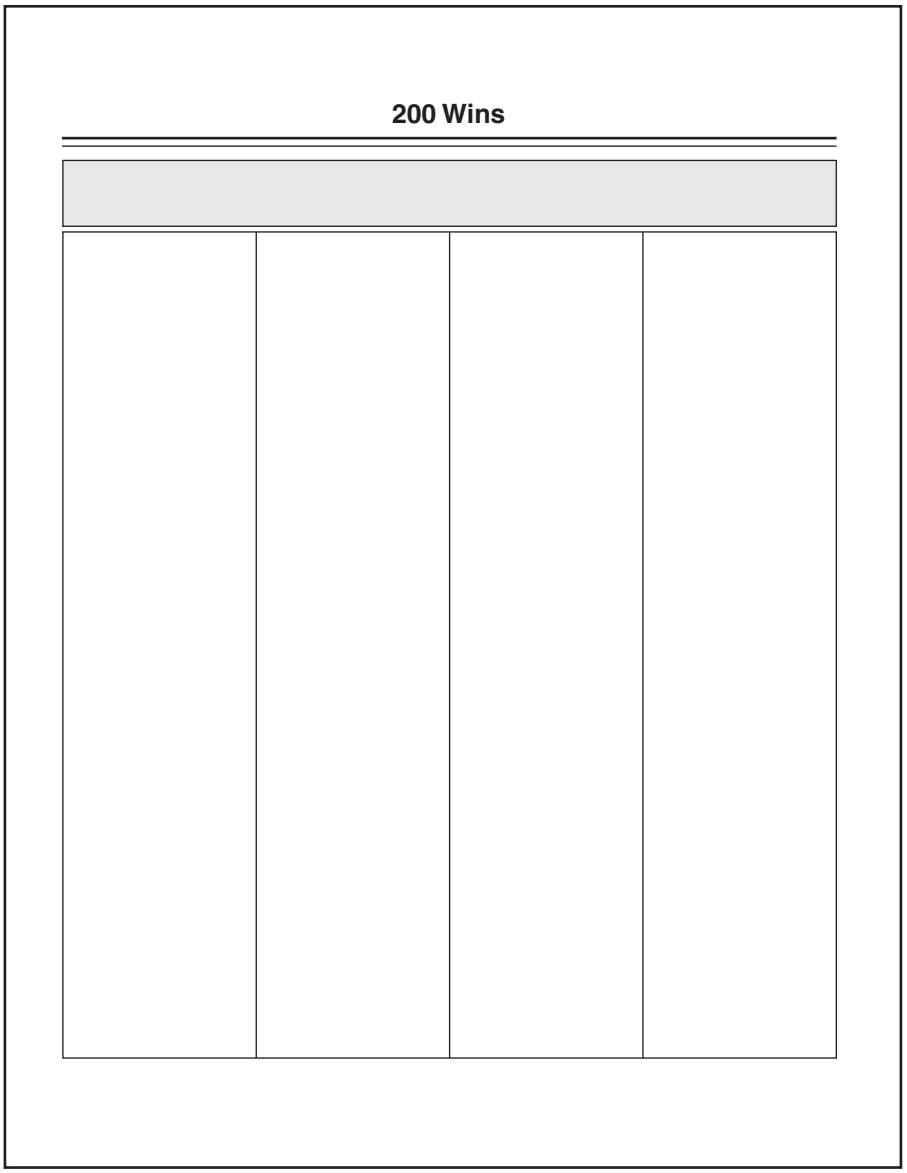
Pencil

Sheet as displayed in Figure 5.4

Two number cubes (one cube with the numbers 1–6; one cube with numbers 0, 1, 2, 7, 8, and 9)

Directions: Four children play, and each child is responsible for one column on her sheet. The names of the players are written on the top of each player's sheet. All players begin with zero points, or with some specified value. The first player

FIGURE 5.4



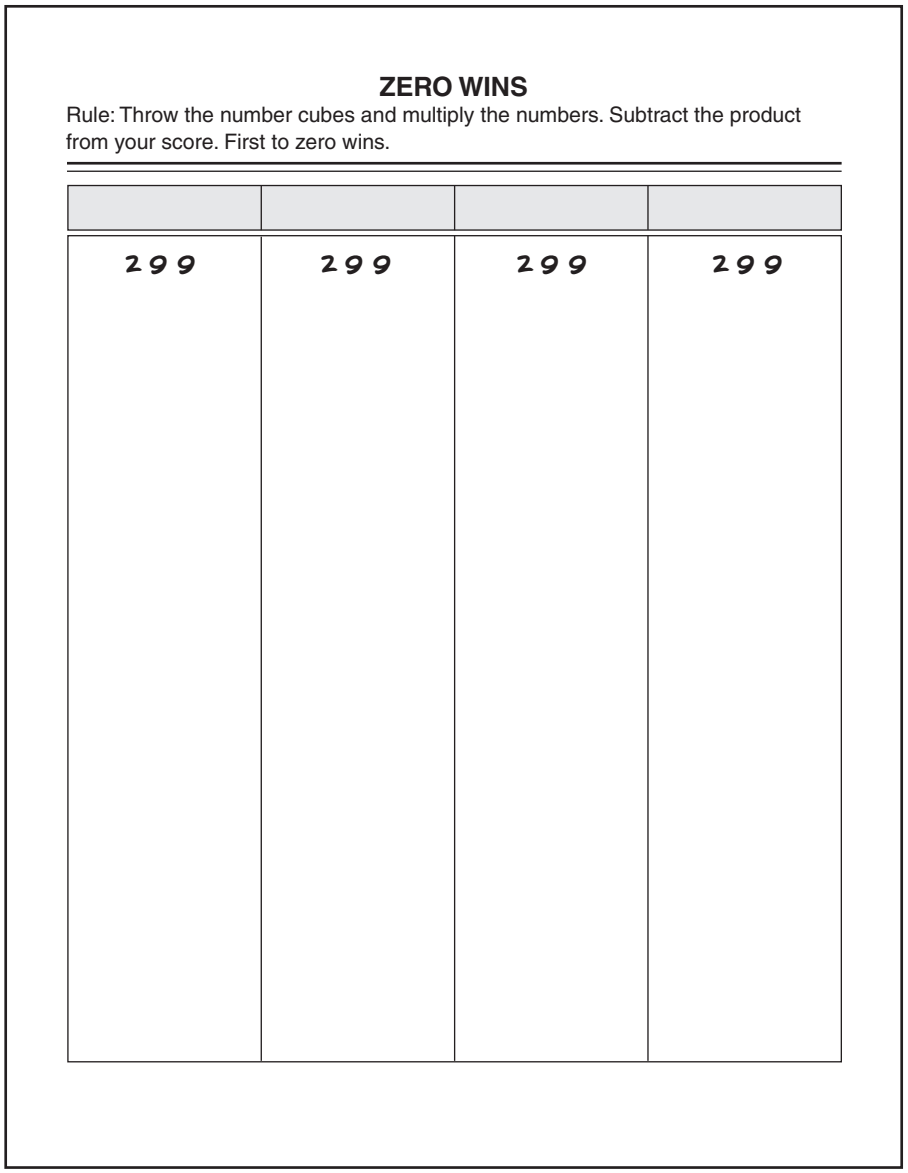
throws and *must say* the product. All players add the number to that player's score on their sheet. The next player then throws, and play continues. If a child thinks the player's product is wrong, she can challenge. If the product was correct, the player receives 10 more points. If the product was incorrect, the players receive no points added to their score. The child who challenged receives 5 points if the challenge is correct. The first player to get 200 wins.

ACTIVITY: Zero Wins

This activity is the same as '200 Wins," except that each player starts with 199 points. Each player obtains a multiplication value as before, then subtracts the value from his score. The first person to zero wins! See Figure 5.5.

Objective: Practice multiplication facts and maintain column subtraction skills

FIGURE 5.5



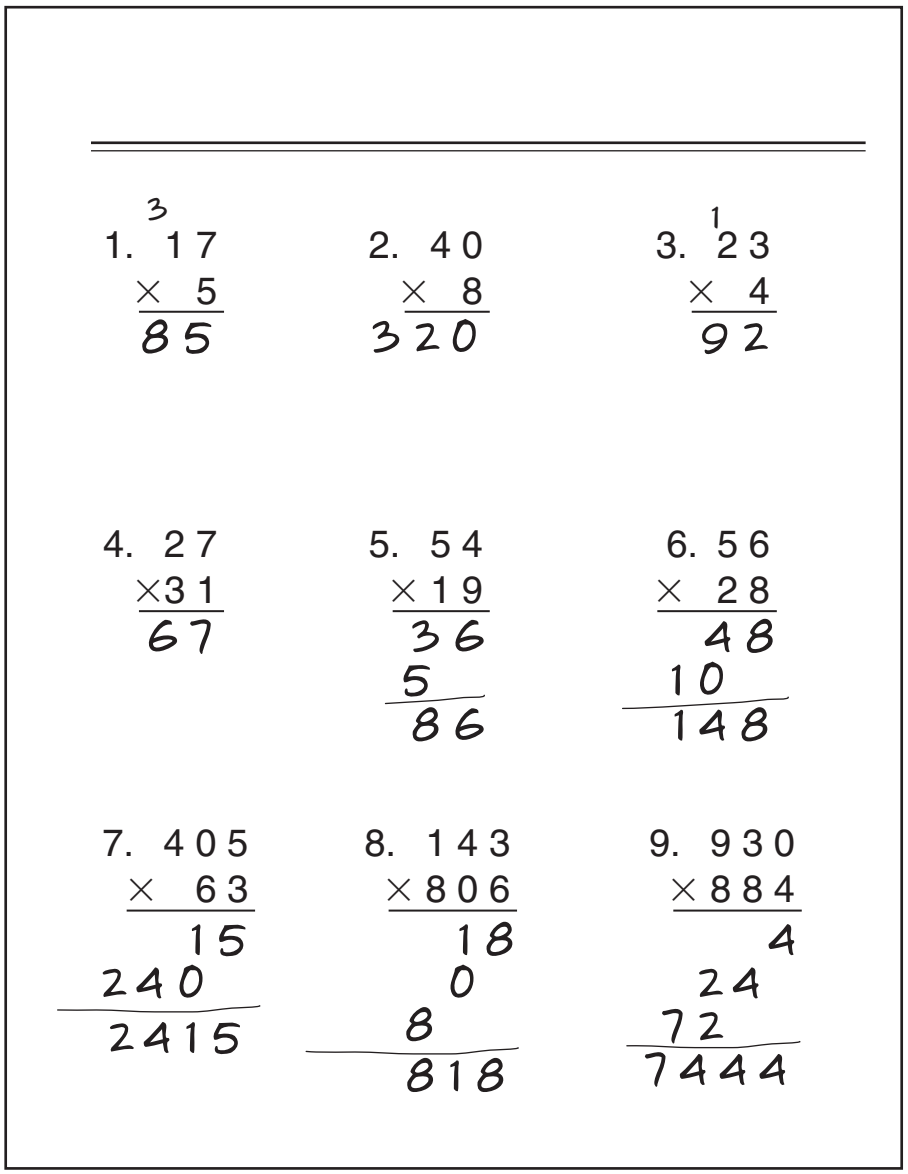
ZERO WINS

Rule: Throw the number cubes and multiply the numbers. Subtract the product from your score. First to zero wins.

299	299	299	299

Discussion Questions

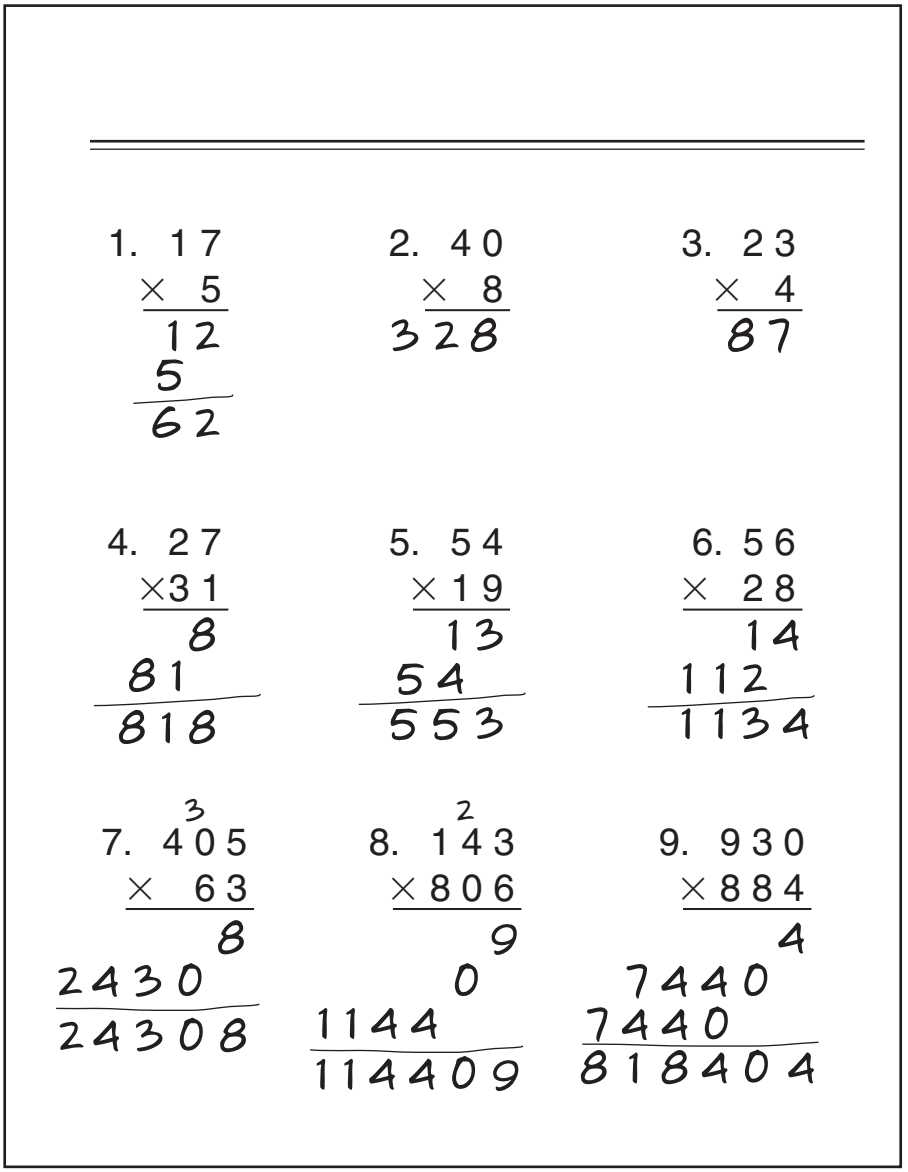
1. Discuss your understanding of the distinction between conceptual errors and procedural errors in multiplication.
2. The following worksheet shows Student Work Sample Error IV for Alberto:



a. Score Student Work Sample Error IV for Alberto. Identify his strengths and error patterns. Complete a DAS for Alberto.

b. Complete an MIP.

3. The following worksheet shows Student Work Sample Error V for Alberto:



   a. Score Student Work Sample Error V for Alberto. Identify his strengths and error patterns. Complete a DAS for Alberto.

   b. Complete an MIP.

4. Design a multiplication activity to provide practice in multiplication facts. Make the activity suitable for a short period of practice for a group of four children.

5. Design a question that would focus on the conceptual understanding of multiplication. The question should help you know if the student understands why multiplication by the tens digit results in all the partial products being groups of tens.

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